Radio Relay

Radio relay is a possible alternative to coaxial cable as a means of providing long-distance television, telephone, or other communication forms. Experiments are under way in this connection as a part of which a full-scale radio-relay system is being constructed between New York and Boston. This system will operate principally in the 4000-Mc. range and will use seven intermediate repeater points. If such a system should prove successful and be capable of being operated at a reasonable cost, there is a possibility that the future will see extensive use of radio relay in long-distance intercity communication. It would be expected that, as in the case of coaxial, the broad frequency bands provided might be utilized for television, telephone, or other types of communication.

Looking Ahead

Past experience has shown a continued trend toward the use of wider and wider frequency bands for communication purposes. Equipment is now under development for use with coaxial cables which will provide wider bands; for example, a 7-Mc. band capable of being used in furnishing an effective 4-Mc. television circuit together with 480 telephone circuits.

For more than a decade the Bell System has conducted research work on a system of transmission in which super-high-frequency waves are guided through hollow pipes. The technique for generation, amplification, and control of the very-high frequencies used in the wave-guide system may also be employed when these frequencies are used for microwave radio. The relative extent to which guided waves or radio beams in space will be used in the future cannot accurately be predicted at this time. Wave guides have the advantage of avoiding some of the possible sources of interference in radio, but they do require the construction of the guiding structure over the route used.

Just how any of these future possibilities will develop, or what other arrangements might be introduced, can not be foretold with any certainty, but it seems clear that the frontiers of broad-band frequency transmission have ample room in which to move forward.

Mutual Impedance Between Vertical Antennas of Unequal Heights*

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***Summary*—An expression is derived for the resistive and reactive components of the mutual impedance between vertical antennas of unequal heights, located above a perfectly conducting ground. Mutual-impedance curves for typical combinations of antenna heights are plotted for spacings between 0.1 and 1.0 wavelength.**

I. Introduction

To THE DESIGNER of power-distribution apparatus for directional antenna arrays, the evaluation of mutual impedance between elements of the array is the first step in a series of calculations leading eventually to the determination of all system parameters. Sufficient data exist in the literature to accommodate the usual problem in which all radiating elements are equal in height, but for the occasional instance involving radiators of unequal heights no mutual-impedance data are available. It is the purpose of this paper to derive an expression from which may be calculated the mutual impedance between radiators of unequal heights, mounted vertically above a perfectly conducting earth, and thus to fill in a gap which in some cases hinders the proper design of antenna-phasing networks.

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II. Formulation of Integral

In Fig. 1, two antennas of heights l_1 and l_2 are shown separated by a distance d. The current at the base of

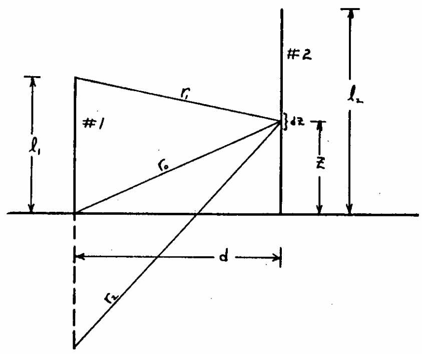

Fig. 1—Two antennas of unequal height above a perfectly conducting earth.

antenna *1* is I_{01}, while that at any distance z above ground is I_{z1}. Similarly, the current at the base of antenna *2* is I_{02}, while that at any distance z above ground is I_{z2}. The term E_{z21} represents the vertical component of electric field at a point z on antenna 2 due to currents in antenna 1, and V_{21} represents the voltage appearing at the base of antenna 2 due to antenna 1.

By application of the reciprocity theorem to the currents and voltages in antenna 2, one can write:

$$V_{21} = \int_0^{l_2} \frac{E_{z21} I_{z2}}{I_{02}} dz. \quad (1)$$

Since the antenna currents are assumed to be sinusoidally distributed, I_{z2} becomes:

$$I_{z2} = \frac{I_{02} \sin \beta(l_2 - z)}{\sin \beta l_2} \quad (2)$$

which, inserted in (1), gives

$$V_{21} = \int_0^{l_2} \frac{E_{z21} \sin \beta(l_2 - z)}{\sin \beta l_2} dz. \quad (3)$$

The mutual impedance, referred to the bases of both antennas, is

$$Z_{21} = -\frac{V_{21}}{I_{01}} = -\int_0^{l_2} \frac{E_{z21} \sin \beta(l_2 - z)}{I_{01} \sin \beta l_2} dz. \quad (4)$$

It is convenient to write (4) in terms of exponentials:

$$Z_{21} = -\int_0^{l_2} \frac{E_{z21}\left[e^{j\beta(l_2-z)} - e^{-j\beta(l_2-z)}\right]}{2jI_{01} \sin \beta l_2} dz. \quad (5)$$

Brown's[1] expression for the vertical component of electric field E_{z21} may now be introduced.

$$E_{z21} = \frac{-j30 I_{01} e^{j\omega t}}{\sin \beta l_1} \left\{ \frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} - 2\frac{e^{-j\beta r_0}}{r_0} \cos \beta l_1 \right\}. \quad (6)$$

Inserting (6) into (5) and dropping the time-variant $e^{j\omega t}$, the mutual impedance is found to be the sum of six integrals.

$$z_{21} = \frac{15}{\sin \beta l_1 \sin \beta l_2} \left\{ \int_0^{l_2} \frac{e^{-j\beta(r_1 - l_2 + z)}}{r_1} dz - \int_0^{l_2} \frac{e^{-j\beta(r_1 + l_2 - z)}}{r_1} dz + \int_0^{l_2} \frac{e^{-j\beta(r_2 - l_2 + z)}}{r_2} dz - \int_0^{l_2} \frac{e^{-j\beta(r_2 + l_2 - z)}}{r_2} dz - 2\int_0^{l_2} \frac{e^{-j\beta(r_0 - l_2 + z)}}{r_0} dz \cos \beta l_1 + 2\int_0^{l_2} \frac{e^{-j\beta(r_0 + l_2 - z)}}{r_0} dz \cos \beta l_1 \right\}. \quad (7)$$

III. Evaluation of Integrals

From the geometry of Fig. 1, the following expressions are apparent:

$$r_1 = \sqrt{d^2 + (l_1 - z)^2} \quad (8)$$

$$r_2 = \sqrt{d^2 + (l_1 + z)^2} \quad (9)$$

$$r_0 = \sqrt{d^2 + z^2}. \quad (10)$$

On differentiating (8), (9), and (10), one obtains

$$\frac{dz}{r_1} = \frac{dr_1}{z - l_1} \quad (11)$$

$$\frac{dz}{r_2} = \frac{dr_2}{z + l_1} \quad (12)$$

$$\frac{dz}{r_0} = \frac{dr_0}{z}. \quad (13)$$

For convenience, let the sum and difference of the radiator lengths be given by

$$\Delta = l_2 - l_1 \quad (14)$$

$$L = l_2 + l_1. \quad (15)$$

By making suitable changes of variable, each integral will now be reduced to the form

$$\int_{u_0}^{u_1} \frac{e^{-ju}}{u} du = [Ci(u_1) - Ci(u_0)] + j[Si(u_0) - Si(u_1)] \quad (16)$$

in which $Ci(u)$ and $Si(u)$ are the cosine and sine integrals, respectively. Tables of these integral functions are available for values of the argument ranging from 0 to 100, the best available compilation having been published in 1940 under WPA sponsorship.[2]

In the first integral, let

$$u = \beta(r_1 - l_1 + z). \quad (17)$$

Then, from (11),

$$du = \beta(dr_1 + dz) = \beta\left(\frac{z - l_1 + r_1}{r_1}\right)dz = \frac{udz}{r_1}$$

and

$$\frac{du}{u} = \frac{dz}{r_1}. \quad (18)$$

New limits of integration u_0 and u_1 are obtained by letting $z = 0$ and $z = l_2$, respectively, in (17).

$$u_0 = \beta[\sqrt{d^2 + l_1^2} - l_1]$$
$$u_1 = \beta[\sqrt{d^2 + \Delta^2} + \Delta]. \quad (19)$$

The first integral in (7) then becomes

$$\int_0^{l_2} \frac{e^{-j\beta(r_1 - l_2 + z)}}{r_1} dz = \int_{u_0}^{u_1} \frac{e^{-j(u + \beta l_1 - \beta l_2)}}{u} du = e^{j\beta\Delta} \int_{u_0}^{u_1} \frac{e^{-ju}}{u} du. \quad (20)$$

In the second integral, let

$$v = \beta(r_1 + l_1 - z). \quad (21)$$

Differentiating and using (11), one obtains

$$\frac{dv}{v} = -\frac{dz}{r_1}. \quad (22)$$

[1] G. H. Brown, "Directional antennas," Proc. I.R.E., vol. 25, pp. 81–145; January, 1937.

[2] "Tables of Sine, Cosine, and Exponential integrals," Federal Works Agency, Work Projects Administration, 1940.

The new limits of integration are

$$v_0 = \beta[\sqrt{d^2 + l_1^2} + l_1]$$
$$v_1 = \beta[\sqrt{d^2 + \Delta^2} - \Delta]. \quad (23)$$

The second integral in (7) then becomes

$$-\int_0^{l_2} \frac{e^{-i\beta(r_1+l_2-z)}}{r_1} dz = e^{-i\beta\Delta} \int_{v_0}^{v_1} \frac{e^{-iv}}{v} dv. \quad (24)$$

The remaining four integrals are transformed in similar fashion.

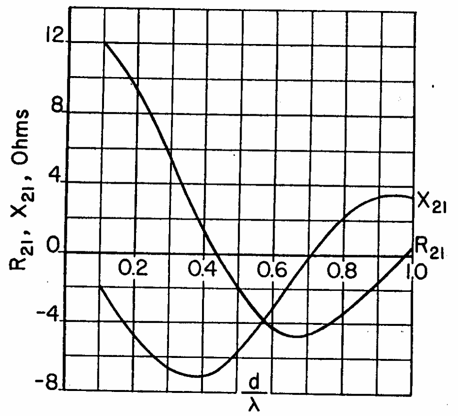

Fig. 2—Resistive and reactive components of mutual impedance between antennas of heights of 40 and 90 degrees.

Changes of variable are made as follows:

$$w = \beta(r_2 + l_1 + z)$$
$$x = \beta(r_2 - l_1 - z)$$
$$y = \beta(r_0 + z) \quad (25)$$
$$s = \beta(r_0 - z).$$

The limits of integration are

$$w_0 = \beta[\sqrt{d^2 + l_1^2} + l_1] = v_0$$
$$w_1 = \beta[\sqrt{d^2 + L^2} + L]$$
$$x_0 = \beta[\sqrt{d^2 + l_1^2} - l_1] = u_0$$
$$x_1 = \beta[\sqrt{d^2 + L^2} - L]$$
$$y_0 = \beta[d] \quad (26)$$
$$y_1 = \beta[\sqrt{d^2 + l_2^2} + l_2]$$
$$s_0 = \beta[d] = y_0$$
$$s_1 = \beta[\sqrt{d^2 + l_2^2} - l_2].$$

After undergoing the transformations indicated, (7) takes the form

$$z_{21} = \frac{15}{\sin \beta l_1 \sin \beta l_2} \left\{ e^{i\beta} \int_{u_0}^{u_1} \frac{e^{-iu}}{u} du + e^{-i\beta\Delta} \int_{v_0}^{v_1} \frac{e^{-iv}}{v} dv + e^{i\beta L} \int_{v_0}^{w_1} \frac{e^{-iw}}{w} dw \right.$$
$$+ e^{-i\beta L} \int_{u_0}^{x_1} \frac{e^{-ix}}{x} dx - 2 \cos \beta l_1 e^{i\beta l_2} \int_{y_0}^{y_1} \frac{e^{-iy}}{y} dy$$
$$\left. - 2 \cos \beta l_1 e^{-i\beta l_2} \int_{y_0}^{s_1} \frac{e^{-is}}{s} ds \right\}. \quad (27)$$

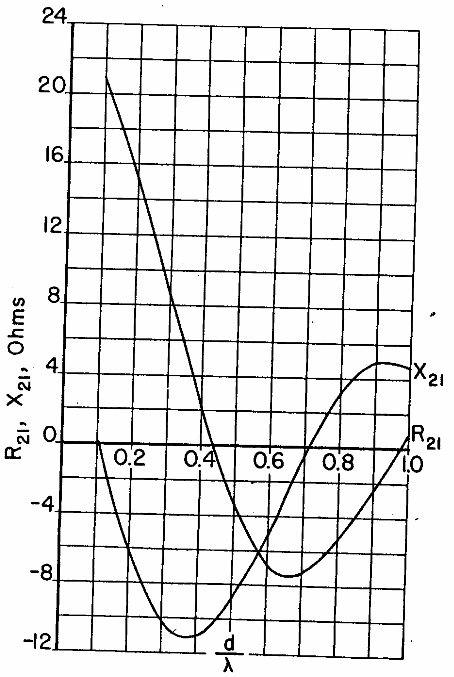

Fig. 3—Resistive and reactive components of mutual impedance between antennas of heights of 60 and 90 degrees.

The final answer is obtained by inserting the cosine and sine integrals of (16) into (27).

$$R_{21} = \frac{15}{\sin \beta l_1 \sin \beta l_2} \{ \cos \beta\Delta [Ci(u_1) - Ci(u_0)$$
$$+ Ci(v_1) - Ci(v_0) + 2Ci(y_0) - Ci(y_1) - Ci(s_1)]$$
$$+ \sin \beta\Delta [Si(u_1) - Si(u_0) + Si(v_0)$$
$$- Si(v_1) - Si(y_1) + Si(s_1)]$$
$$+ \cos \beta L [Ci(w_1) - Ci(v_0) + Ci(x_1) - Ci(u_0)$$
$$+ 2Ci(y_0) - Ci(y_1) - Ci(s_1)]$$
$$+ \sin \beta L [Si(w_1) - Si(v_0) + Si(u_0)$$
$$- Si(x_1) - Si(y_1) + Si(s_1)]\} \quad (28)$$

$$X_{21} = \frac{15}{\sin \beta l_1 \sin \beta l_2} \{ \cos \beta\Delta [Si(u_0) - Si(u_1)$$
$$+ Si(v_0) - Si(v_1) + Si(y_1) - 2Si(y_0) + Si(s_1)]$$

$$
\begin{aligned}
&+ \sin \beta\Delta[Ci(u_1) - Ci(u_0) + Ci(v_0) \\
&- Ci(v_1) - Ci(y_1) + Ci(s_1)] \\
&+ \cos \beta L[Si(v_0) - Si(w_1) + Si(u_0) - Si(x_1) \\
&+ Si(y_1) - 2Si(y_0) + Si(s_1)] \\
&+ \sin \beta L[Ci(w_1) - Ci(v_0) + Ci(u_0) \\
&- Ci(x_1) - Ci(y_1) + Ci(s_1)]\}. \qquad (29)
\end{aligned}
$$

Examination of (28) and (29) reveals that, to obtain X_{21} from R_{21}, it is necessary only to replace $Ci(p)$ by $-Si(p)$, and $Si(q)$ by $Ci(q)$.

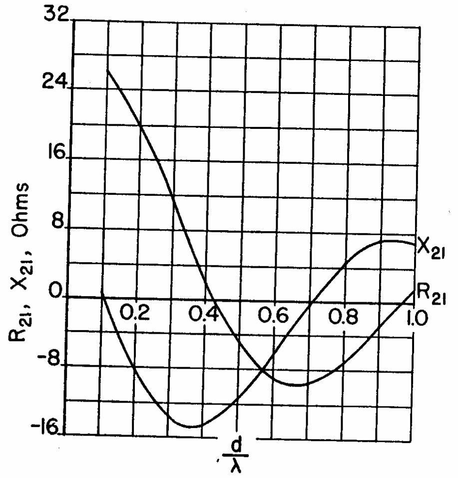

Fig. 4—Resistive and reactive components of mutual impedance between antennas of heights of 75 and 90 degrees.

IV. Special Cases

It is easily shown that when the radiators are equal in height, $l_1 = l_2$ and (28) and (29) become

$$
\begin{aligned}
R_{21} = \frac{15}{\sin^2 \beta l}\, &\{4Ci(u_1) - 2Ci(u_0) - 2Ci(v_0) \\
&+ \cos 2\beta l[Ci(w_1) - 2Ci(v_0) + Ci(x_1) \\
&- 2Ci(u_0) + 2Ci(u_1)] \\
&+ \sin 2\beta l[Si(w_1) - 2Si(v_0) \\
&- Si(x_1) + 2Si(u_0)]\} \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
X_{21} = \frac{15}{\sin^2 \beta l}\, &\{- 4Si(u_1) + 2Si(u_0) + 2Si(v_0) \\
&+ \cos 2\beta l[- Si(w_1) + 2Si(v_0) - Si(x_1) \\
&+ 2Si(u_0) - 2Si(u_1)] \\
&+ \sin 2\beta l[Ci(w_1) - 2Ci(v_0) \\
&- Ci(x_1) + 2Ci(u_0)]\}. \qquad (31)
\end{aligned}
$$

Considerable simplification of (28) and (29) results if one of the antennas (say l_2) is a quarter-wave in height. In this case, the resistive and reactive components of z_{21} become

$$
\begin{aligned}
R_{21} = 15\{&Ci(u_1) + Ci(v_1) - Ci(w_1) - Ci(x_1) \\
&+ \cot \beta l_1[Si(u_1) - Si(v_1) - 2Si(y_1) \\
&+ 2Si(s_1) + Si(w_1) - Si(x_1)]\} \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
X_{21} = 15\{&Si(w_1) + Si(x_1) - Si(u_1) - Si(v_1) \\
&+ \cot \beta l_1[Ci(u_1) - Ci(v_1) - 2Ci(y_1) \\
&+ 2Ci(s_1) + Ci(w_1) - Ci(x_1)]\}. \qquad (33)
\end{aligned}
$$

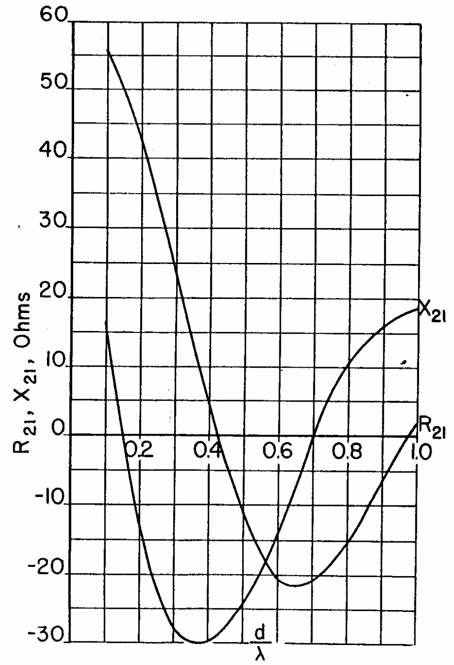

Fig. 5—Resistive and reactive components of mutual impedance between antennas of heights of 120 and 90 degrees.

Figs. 2, 3, 4, and 5 are plots of (32) and (33) for values of l_1 corresponding to angular heights of 40, 60, 75, and 120 degrees, respectively.

V. Conclusion

As is the case for previously available data on mutual impedance with equal heights, the expressions presented here are only approximate, because sinusoidal current distributions are assumed in both members. However, the present calculations are no less accurate, and should prove useful to the same extent as previously published information.

VI. Acknowledgments

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