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SUMMARY

OBJECTIVES

Examine the design parameters for systems of multiwire cages carrying high voltages above ground in order to minimize the maximum electric field on the surface of the wires. For the two-wire case, compare the following methods of calculation: (1) a simple geometrical theory formula (SGT); (2) a complete geometric theory (CGT); (3) an infinite series of images ("exact solution" for given constraints); and (4) a computer-based method of moments (MOM) solution. Calculate the MOM results for cages with 2 through 6 wires, and for 12 wires.

RESULTS

The MOM accurately matches exact solution results for the two-wire case. The CGT formula reproduces the results of the more exact solutions fairly well. For 3 or more wires, a comparison of results of MOM, SGT, and CGT suggests a modification to the SGT called modified simple geometric theory (MSGT). The MSGT requires few calculations, can be applied to cages with any number of wires, and reproduces the MOM results surprisingly well for all the cases.

The results show that there is an optimum wire spacing to minimize surface electric field. This spacing depends upon the number of wires in the cage, their diameter, and height above ground. The optimum dimensions have been tabulated and design curves are given for cages using 2, 3, 4, 5, 6, and 12 wires.

The concept of electric field equivalent radius is found to be extremely useful. Application of this concept leads to a simplified design procedure. Two design procedures are given, one based on minimum wire spacing for a given wire diameter, and the second based on minimizing the total wire cross section of the cage. These procedures are presented with three examples that include antenna and transmission line applications. Application to cylindrical geometry is considered as is the effect of shielding from other objects in the vicinity of the cage.

The modified simple geometrical formula and the design procedures given are useful for design of high-voltage transmission lines and high-power transmitting antennas.

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1.0 INTRODUCTION

1.1 GENERAL

The application of multiwire cages, sometimes called bundles, to high-voltage antennas and for power distribution goes back many years. The early workers in this field found that several wires together could carry more voltage than a single wire (Clark, 1932; Temoshok, 1948; Adams, 1955; Miller, 1956). In the past, approximate formulas have been used to calculate the surface electric fields on the wires in a cage. By using modern computer techniques to generate more exact data, we have discovered how to modify the simple formula to provide more accurate results.

For a fixed-wire diameter, a fixed number of wires, and a fixed cage height, there is an optimum spacing having minimum surface gradient.

1.2 OBJECTIVE

This report will present the modified formula for the calculation of surface electric fields on multiwire cages, and, two optimum design techniques for multiwire cages.

1.3 BACKGROUND

Many transmitting antennas in the lower frequency bands are of the top-loaded monopole type. Often, the top load is composed of cables suspended from towers, or across valleys, etc. These cables can carry high voltages and, in many ways, exhibit similar design problems as the high-voltage power transmission lines.

If the antenna is small with respect to wavelength, than the maximum power it can radiate is proportional to top-load area squared, frequency to the fourth power, and maximum voltage squared (Hansen, 1990; Watt, 1967). The maximum voltage limit for an antenna is determined by voltage breakdown on insulators, and/or by corona formation on the structure. Conductor selection for the various portions of the antenna is one of the most important design considerations. The conductor size for high-voltage applications is often established by corona performance rather than by current carrying capacity or structural considerations.

1.3.1 Corona

Corona forms when the surface electric field (potential gradient) exceeds the breakdown strength of air. Undesirable effects of corona include (1) radio noise, (2) audible noise, and (3) power loss. In addition, chemical byproducts of corona can increase corrosion rates. Considerable study of these effects has been done by the power distribution community (Peek, 1929; EPRI, 1987). A further consideration for high-Q VLF/LF antennas is that a significant amount of corona will change the effective antenna capacitance and, thus, make it difficult to maintain tuning.

The voltage at which corona first appears is called the onset voltage V_0. The larger the wire size, the larger the V_0. V_0 is considerably reduced for wet conditions. Economy dictates that power distribution lines be designed to operate with some corona, especially during inclement weather.

The effects of corona are proportional to frequency. For VLF, and even more for LF, having any portion of the antenna in corona is undesirable. At these frequencies, even a small segment of wire in corona can dissipate a large amount of power (Watt & Hansen, 1992).

The onset of corona correlates with the maximum surface electric field (Peek, 1929; Comber et al., 1987). For this reason, the surface electric field (E-field often called voltage gradient or gradient) is used to evaluate corona performance. The surface gradient resulting in corona onset is called the critical onset gradient E_0. The critical gradient for a single conductor at 60 Hz has been the subject of considerable study (Peek, 1929; EPRI, 1987). Peek developed an empirical formula for E_0 versus wire diameter at 60 Hz for dry single conductor wires:

$$E_0 = 2.107\ d\{1 + 0.952/(d\ a)^{0.5}\}\ \text{kV/mm rms},$$

where

d = relative air density (1 @ STP), and
a = wire radius (mm).

This can be applied to the calculation of the onset of corona by using a surface roughness factor M and a frequency factor F. The surface roughness factor takes into account that the local gradients are higher than the average gradient.

Some study has taken place of the critical gradient in the VLF and LF bands. The critical gradient at VLF and LF for dry wires follows the same law as at 60 Hz, with a slight reduction for frequency (Smith, 1963, 1985; Watt, 1967).

The critical gradient depends on many variables. Among the important parameters are frequency, atmospheric conditions (e.g., pressure, temperature, and humidity), surface conditions (wet or dry, oil or other contaminants), and surface roughness. In particular, for both VLF and LF, the critical gradient is considerably reduced for wet conditions.

Wire diameter has a strong influence on the critical gradient, with smaller wires having larger critical gradients, because the electric field falls off more rapidly with distance from the surface for smaller wires. For ionization (corona) to occur, electrons from the surface must reach the critical velocity for ionization within a few collision distances. This critical velocity depends on the type of molecules involved in the collisions. Electron velocity is proportional to the integral of electric field times distance from the surface. Because of the rapid decrease in electric field near the surface of smaller wires, the surface field must be higher to support corona. However, since the surface electric field is inversely proportional to radius, the voltage required to give this critical gradient is less for smaller wires. Thus, smaller wires go into corona at lower voltages, but with larger surface gradients than larger wires.

The variation of critical gradient with radius is strongly influenced by surface conditions, polarity, and frequency. For example, large wires, when wet, have a lower critical gradient than when dry, while small wires can exhibit the opposite performance.

The wire diameter used for antennas and transmission lines usually exceeds 1/2 inch. Wires of this diameter have a lower critical gradient when wet. The wet value is used as the design parameter for outside applications.

A complete discussion of this phenomenon is beyond the scope of this paper. Experiments have shown that corona effects can be conveniently expressed in terms of the maximum conductor gradient. For this reason, the design specifications for corona performance are stated in terms of maximum allowable surface gradient. For simplicity, a single design value of surface gradient is often specified. This is usually acceptable for the larger diameter wires typically used for transmission lines and antennas, because the critical gradient versus diameter does not change too much over the range of the wire diameters. For VLF/LF antennas near sea level with wires in the vicinity of 1-inch-diameter, common design criteria use maximum surface gradients in the range of 0.7 to 0.9 kV/mm RMS (Smith, 1985). For 60 Hz applications, a design limit for wet conditions used by some power line designers is 1.2 kV/mm[1] (EPRI, 1987).

1.3.2 Surface Gradient Calculations

Calculation of the exact gradient at every point on the surface of cables is complicated by the fact that the cables, often multiply stranded, usually are not smooth. In addition, the geometry generally causes the electric gradient to be nonuniform around the wire periphery.

To simplify the calculation of the surface electric field for corona performance determination, the conductors are usually treated as smooth cylinders. Correction factors are then used for surface roughness and frequency effects. For stranded wires, the typical correction factor for surface roughness (M) varies between 0.83 and 0.87 (Chaudhari, see Miller, 1956).

The effect of frequency on E_0 is small at VLF and LF. The frequency effect factor F has been found empirically to be on the order of 0.90 at VLF (Watt & Hansen, 1992).

To evaluate the corona performance of multiple wire cages, it is necessary to calculate the value of the surface electric field. In the past, several authors have examined this problem (Clark, 1932; Temoshok, 1948; Adams, 1955; Miller, 1956; Comber et al., 1987; Doyle et al., 1987). Miller (1956) has some measured and calculated 60 Hz results. Although some recent efforts to calculate the gradients on cages make use of computer techniques, all involve some approximations. For cases that can be approximated by a single isolated cage above ground, a simple formula had previously been developed. In the following sections, this formula, as well as some other formulas, will be developed and compared with computer results calculated by an "essentially exact" method-of-moment technique.

1. Personal communication with J. Bainbridge, Transmission Line Engineering Section, San Diego Gas and Electric Corporation, September 1987.

1.4 CAGES

1.4.1 Definitions/Geometry

The cage geometry is illustrated in figure 1 for the general n wire cage. The cage center height above ground is (h), the radius of the cage is (b), and the radius of the individual wires is (a). MKS units are used.

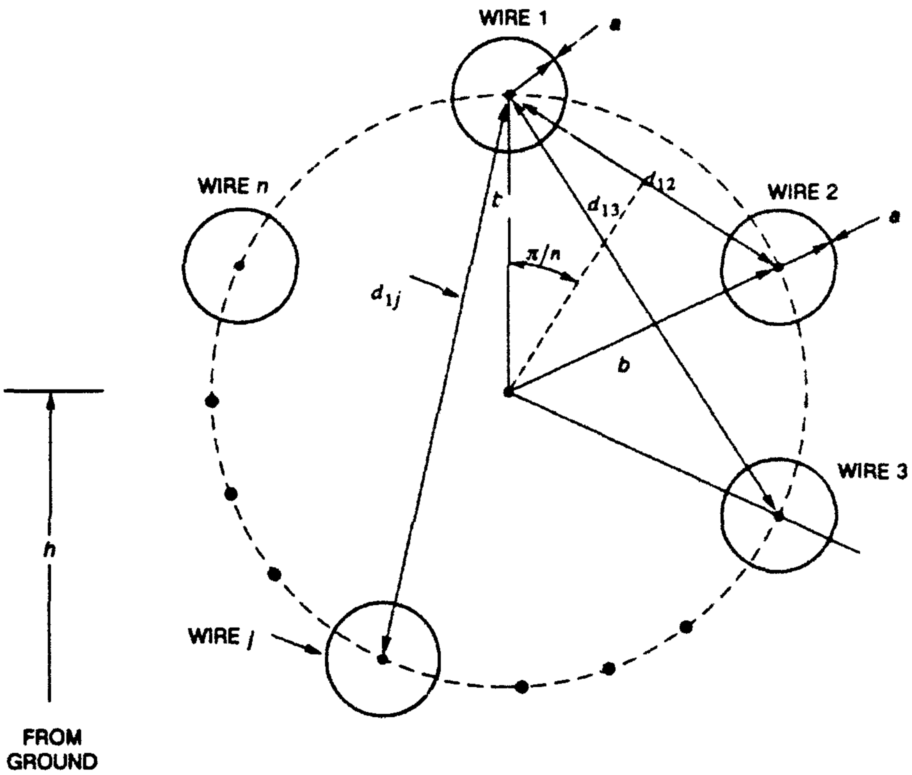

Figure 1. n-wire cage geometry.

1.4.2 Assumptions

Several assumptions are operative:

(1) The wires are parallel to each other and to ground.

(2) The wires are perfectly smooth circular cylinders.

(3) The wires and the ground are perfect conductors.

(4) All wires are at the same potential V.

(5) The wires all have the same radius a.

(6) The wires are uniformly separated on a circle of radius b.

(7) The wires are much farther above ground than the cage radius b (i.e., $h >> b$).

The last assumption results in some simplifications:

(1) The charge is the same on all wires.

(2) The electric fields in the vicinity of the wires are not affected by the presence of the ground (note: charge magnitude is affected).

1.5 THEORY

1.5.1 Matrix Equations

A. General Equations

The charge on each wire is assumed to be a filament located at the center of the wire. This is a valid approximation if the wires are separated far from ground and from each other relative to their diameter. As the wires come closer together, this approximation breaks down. This effect will be examined in the following section and a correction term developed.

Following the notation and method of Miller (1956), assuming wires with the charge at the center, the Maxwell potentials define the relationship between voltage and charge as follows:

Self Potential

$$P_{ii} = \ln(2h/a_i)/(2\pi\epsilon_0) ,$$

Mutual Potential

$$P_{ij} = \ln(D_{ij}/d_{ij})/(2\pi\epsilon_0) ,$$

where a_i is the radius of the ith wire, d_{ij} is the distance between wires i and j, while D_{ij}, is the distance between wire i and the image of wire j reflected in the ground plane. ϵ_0 is the permittivity of free space.

The relationships between voltage and charge are given by

$$
\begin{aligned}
V_1 &= q_1P_{11} + q_2P_{12} + \cdots + q_nP_{1n} , \\
V_2 &= q_1P_{21} + q_2P_{22} + \cdots + q_nP_{2n} , \\
&\bullet \\
&\bullet \\
&\bullet \\
V_n &= q_1P_{n1} + q_2P_{n2} + \cdots + q_nP_{nn} ,
\end{aligned}
$$

where q_i is the charge and V_i the voltage on the ith wire. In matrix notation, the above set of equations is

$$[V] = [P][q] .$$

Here $[q]$ is a column vector.

B. Simplified Equations

The above equations can be solved for the general case of different voltages, wire diameters, and locations. This may be required in some cases, for example, to obtain charge densities on multiphase power lines in proximity to each other and to ground. See, for example, Doyle et al. (1987), as well as Miller (1956), and Adams (1955). However, these equations can be greatly simplified for the case of a single bundle composed of wires all having the same radius (a), at the same potential (V), and located at a large distance above the earth with respect to the bundle dimension (i.e., $h >> b$). Stating this in terms of equations,

$$V_1 = V_2 = \cdots = V_n = V ,$$

$$a_1 = a_2 = \cdots = a_n = a ,$$

and the fact that $h >> b$ leads to the approximation that

$$D_{ij} = 2h = D \quad \text{for all } i \text{ and } j, \text{ and}$$

$$q_1 = q_2 = \cdots = q_n = q .$$

Substituting the above into the matrix equations and using reciprocity ($P_{ij} = P_{ji}$), leads to a single equation for the charge density on the individual wires

$$q = V/(P_{11} + P_{12} + \cdots + P_{1n}) .$$

1.5.2 Capacitance - Equivalent Radius

The total charge on the bundle is

$$Q = nq ,$$

and the total capacitance of the bundle is

$$C = Q/V = n/(P_{11} + P_{12} + \cdots + P_{1n}) .$$

A. Single Wire

For a single wire, the capacitance is simply given by

$$C_1 = 2\pi\epsilon_0/\ln(2h/a) . \tag{1}$$

The electric field at the surface of the single isolated wire obtained using Gauss' law is given by

$$E_1 = q/(2\pi\epsilon_0 a) = V\{1/a\}[1/\ln(2h/a)] . \tag{2}$$

Note the surface electric field formula contains one term related to geometry enclosed by { }, and another term related to capacitance enclosed by [ ].

B. General Case (*n*-Wires)

The general case of n wires uniformly spaced on a circle of radius b is illustrated in figure 1. The spacing between the centers of wires 1 and j (d_{1j}) is

$$d_{1j} = 2b \sin(j\pi/n) \ .$$

Substitution into (1) gives

$$C_n = 2n\pi\epsilon_0/\{\ln(2h/a) + \ln(2h/(2b\sin(\pi/n)) +$$

$$\ln(2h/(b\sin(2\pi/n)) + \cdot\cdot\cdot + \ln(2h/(2b\sin((n-1)\pi/n))\} \ .$$

Combining the log terms in the denominator gives

$$C_n = 2n\pi\epsilon_0/\{\ln((2h)^n/[a(2b)^{(n-1)} \sin(\pi/n) \ \sin(2\pi/n) \cdot\cdot\cdot$$

$$\cdot\cdot\cdot \ \sin(2j\pi/n) \ \cdot\cdot\cdot \ \sin((n-1)\pi/n)])\} .$$

Further reduction using the following identity (appendix A),

$$2\sin(\pi/n) \ 2\sin(2\pi/n) \ \cdot\cdot\cdot \ 2\sin(j\pi/n) \ \cdot\cdot\cdot$$

$$\cdot\cdot\cdot \ 2\sin((n-1)\pi/n) = n \ , \quad \text{gives}$$

$$C_n = 2n\pi\epsilon_0/\{n \ \ln(2h/a')\} \ , \quad \text{or}$$

$$C_n = 2\pi\epsilon_0/\{\ln(2h/a')\} \ . \tag{3}$$

Where $a' = (nab^{(n-1)})^{(1/n)}$ is the cage equivalent radius for capacitance, or the radius of a single wire having the same capacitance per unit length as the cage. Note that equation (3) has the same form as equation (1).

This simple expression for the capacitance per unit length is quite accurate as will be shown in section 2.0. This result is equivalent to one given by Grover (1928). However, he did not obtain the closed form. The closed form expression is included in gradient formulas given by Crary (1932) and Chaudhari (Miller, 1956). The above expression for equivalent radius is the same as the geometric mean radius of the cage (Grover, 1962) and gives the correct inductance in the high-frequency limit. Shelkunoff (1952) has derived the same expression for equivalent radius by a completely different method and shows that it gives the correct value for the transmission line impedance of a multiple wire cage.

C. Cylindrical Geometry

If the cage is located at the center of a cylinder of radius R instead of a height h above the ground, the formula for capacitance uses the same cage equivalent radius a' but will replace $2h$ by R. In section 1.5.3, capacitance is always part of the formula used for calculating the maximum surface electric field on the cages. For cylindrical geometry, use the same formulas but replace $2h$ by R in the capacitance term.

D. Capacitance In The Presence Of Other Wires

1. *Two-Wire Case*

Often, it is desirable to design a multiwire cage for each of two or more widely separated wires carrying a high voltage. The presence of the other wires modifies the capacitance of any particular wire. For certain conditions, the effect of this modification can be accounted for by changing the apparent height of the wire h_a.

The equations for the two-wire case are

$$2\pi\epsilon_0 V_1 = q_1 \ln\left(\frac{2h}{a}\right) + q_2 \ln\left(\frac{D}{S}\right)$$

and

$$2\pi\epsilon_0 V_2 = q_1 \ln\left(\frac{D}{S}\right) + q_2 \ln\left(\frac{2h}{a}\right) ,$$

where D is the slant distance between one wire and the image of the other wire, and S is the distance between wires. If each of the individual wires were caged, then the radius (a) would be replaced by a_{eqC}.

If both wires have the same radius a (or a_{eqC}) and the same voltage (hence, the same charge), then the equations reduce to

$$V = \frac{q}{2\pi\epsilon_0} = \left[\ln\left(2\frac{h}{a}\right) + \ln\left(\frac{D}{S}\right)\right] .$$

The capacitance of each individual wire is given by

$$C = \frac{q}{v} = \frac{2\pi\epsilon_0}{\ln\left(\frac{2h}{a}\frac{D}{S}\right)} .$$

From the above equation, it is clear that the capacitance is the same as that of a single isolated wire at a different height. This new height is defined to be the apparent height h_a and is given by

$$h_a = h\left(\frac{D}{S}\right) .$$

Thus, for this case, each wire can be treated as a single isolated wire at a new apparent height. Note that the assumptions required are that both wires have the same (1) radius a, (2) height above ground, and (3) voltage. For the case of caged wires, a is replaced by a_{eqC}, which must be the same for both wires.

2. *Infinite Wire Grid*

The case of an infinite number of parallel wires separated by a distance S and located at the same height h above the ground can be treated in the same manner. The capacitance per unit length of one wire in the grid is derived in appendix B and is given below:

$$C_\infty = \frac{2\pi\epsilon_0}{\ln\left(\frac{2h}{a}\right) + \ln(X)} ,$$

where X is defined as

$$X = \left(\frac{\sinh\left(\frac{2\pi h}{S}\right)}{\left(\frac{2\pi h}{S}\right)} \right)^{1/2} .$$

This equation is similar to that for the two-wire case. From this equation, it can be seen that the concept of an apparent height is valid for an infinite wire grid. For this case, it is clear that h_a is given by the following,

$$h_a = h \left(\frac{\sinh\left(\frac{2\pi h}{S}\right)}{\left(\frac{2\pi h}{S}\right)} \right)^{1/2} .$$

Note that for cases like those above, where shielding is provided by other wires at the same potential, the apparent height is considerably greater than the actual height.

1.5.3 Surface Electric Field

A. Simple Geometric Theory

The electric field is calculated by superimposing the fields due to the charges on each wire. The field has a maximum on the surface of each wire at the point opposite the cage center. Figure 2 illustrates the method of calculation of the maximum field on one wire by vector addition of the field due to the charge on the wire itself, plus the radial component of the fields due to the charge on the other wires. The field due to the charge on the wire itself is

$$E = q/(2\pi\epsilon_0 a) ,$$

where q is the charge on each individual wire, given by

$$q = C_n V/n .$$

To this must be added the fields, due to the charges on the other wires. The magnitude of these fields can be approximated by calculating them at the point corresponding to the center of the wire in question, as if the wire were not there. The effect of the wire (cylinder) is approximated by assuming that the field is uniform and, therefore, doubles at the point where the field is perpendicular to the surface (Morse & Feshback, 1953). Crary (1932) seems to be the first to have suggested this approximation. The doubling of the field on the surface at that point can also be derived using image theory (section 2, appendix C).

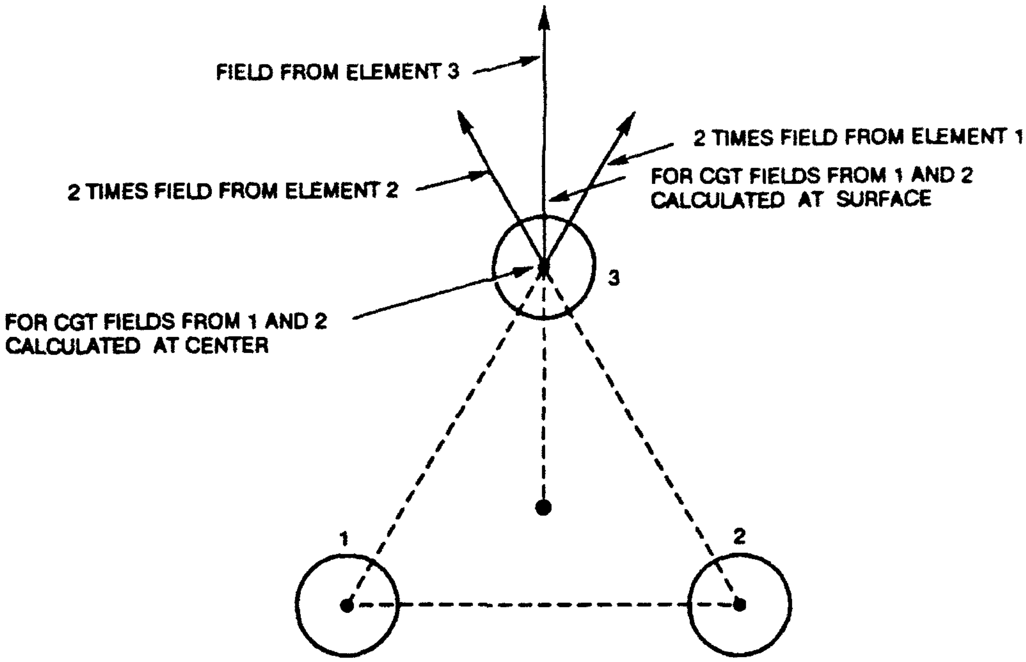

Figure 2. Field calculation.

From symmetry (figure 2), it is seen that, at the location of maximum field on the surface of a wire, the field due to the charge on the other wires will be normal to the surface. If the field were uniform, then the effect of the cylinder on this field can be accounted for by multiplying by a factor of 2. The uniform field approximation is accurate when the wires are well separated, but becomes less accurate when the wires are closer together. The accuracy of this approximation will be examined further in the next section.

From figure 1 and Gauss' law, it can be seen that the field at the location of the center of wire 1 due to the charge on the jth wire, is given by

$$E = q/(2\pi\epsilon_0 d_{ij}) = q/\{2\pi\epsilon_0 2b \, Sin(j\pi/n)\} \quad .$$

To get the component in the direction of the cage radial requires multiplication of the above term by $\cos(\chi_j)$, which cancels out the $\sin(j\pi/n)$ term (figure D-1, appendix D). Thus, the contribution to the radially directed electric field at the location of the center of a wire, due to the charge on each of the other wires, is identical and given by

$$E = q/(4\pi\epsilon_0 b) \ .$$

The maximum surface electric field then is approximated by the sum of the field from the wire itself plus twice the sum of the contributions from the other wires, given by

$$E_n = \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{a} + \frac{n-1}{b}\right\} \ .$$

Substituting for q by using the capacitance formula C_n, gives

$$E_n = \frac{C_n V}{2\pi\epsilon_0}\left(\frac{1}{n}\right)\left\{\frac{1}{a} + \frac{n-1}{b}\right\} .$$

Substituting for C_n gives

$$E_n = \frac{V}{n}\left\{\frac{1}{a} + \frac{n-1}{b}\right\}\left[\frac{1}{\ln(2h/a')}\right] , \tag{4}$$

where { } contains the geometry term, and [ ] the capacitance term. Equations (3) and (4) are called the simple geometric theory (SGT). This formula appears in Chaudhari (Miller, 1956) and Smith (1985).

B. Complete Geometric Theory

The simple geometric formula is based on estimating the field in the vicinity of one cylinder due to all the other cylinders by calculating the field at the center of the wire as if it were not there. This field is then doubled to account for the effect of the cylinder. This would be a valid approach if the field in the vicinity of the cylinder were uniform. This assumption is violated when the wires are in close proximity. From the discussion of the image method in appendix C, it is clear that a better approximation would be to use twice the field calculated at the surface of the cylinder instead of at the center of the cylinder.

The complete geometric theory (CGT) is derived by the same method as the simple geometric theory (SGT) except that the fields due to the charge on the other wires are calculated on the surface of the wire, not at the center of the wire. The derivation is given in detail in appendix D. The simple formula for capacitance (equation 3) is used in this derivation.

The resulting equation for the maximum surface field is similar to SGT but more complicated. The geometric factor for the two-wire cage is

$$\{\,\} = \left\{\frac{1}{a} + \frac{1}{b + a/2}\right\} .$$

The geometric terms for three or more wires are increasingly more complicated. They are derived explicitly in appendix D for the three- and four-wire case. The general expression for the geometric term is a finite summation, which is also given in the appendix.

C. Modified Simple Geometric Theory

Note that for the two-wire case, the only difference between SGT and CGT is that the factor $a/2$ has been added to the denominator of the second term in the geometric factor. This suggests a modification to the SGT formula having a geometric factor of the form

$$\{\,\} = \left\{\frac{1}{a} + \frac{n-1}{b + a/2}\right\} .$$

Replacing the geometric factor in equation (4) with the above factor results in the following equation, which is called the modified simple geometric theory (MSGT):

$$E_n = \frac{V}{n}\left\{\frac{1}{a} + \frac{n-1}{b+a/2}\right\}\left[\frac{1}{\ln(2h/a')}\right] . \tag{5}$$

Appendix D also contains an alternate derivation of the MSGT formula.

As will be shown in section 2.0, the MSGT formula provides a surprisingly good fit to reality for all cases considered.

Note that for the two-wire case, the formulas for the MSGT and CGT are the same.

2.0 COMPARISON WITH MORE EXACT METHODS

2.1 TWO-WIRE CASE

This section compares the results of the SGT and MSGT to several other methods of calculation for two-wire cages.

Substitution of $n = 2$ in equations (3), (4), and (5) gives

$$C_2 = 2\pi\epsilon_0\left[1/\ln\left(\frac{2h}{(2ab)^{(1/2)}}\right)\right] ,$$

$$E_2 = \frac{V}{2}\left\{\frac{1}{a} + \frac{1}{b}\right\}\left[1/\ln\left(\frac{2h}{(2ab)^{(1/2)}}\right)\right] \quad \text{SGT , and}$$

$$E_2 = \frac{V}{2}\left\{\frac{1}{a} + \frac{1}{b+a/2}\right\}\left[1/\ln\left(\frac{2h}{(2ab)^{(1/2)}}\right)\right] \quad \text{MSGT}$$

2.1.1 Charge Offset Method

Appendix E contains a different method for the derivation of capacitance and surface gradient for the two-wire case. This derivation uses a single line charge near, but offset from, the center of each wire. For the case of two wires with equal charge of opposite sign, this is an exact solution. However, for the multiwire cage geometry that we are using, where the wires are at the same potential, this method is approximate.

For this case, the offset is adjusted to give equal potential at the wire surface, at two points located on opposite sides of the wire. It is interesting to note that the offset cancels out implicitly, and the resulting formulas for capacitance and surface gradient are exactly the same as the SGT.

2.1.2 Image Theory

The electric field in the vicinity of a bundle of charged conductors can be calculated by using the method of multiple image charges (EPRI, 1987). A charge is assumed on one cylinder and imaged on the other cylinder. The image charge is then imaged back into the first cylinder, etc. This leads to an infinite series solution, which is, in practice, truncated. Care must be taken to construct a series that converges rapidly. The secret is to develop a series of images using pairs of charges forming dipoles, that have no net charge, and ever decreasing dipole moment (appendix C). The series converges to within 0.1% with 40 terms when the two wires are touching, and much faster as the two-wires are spaced farther apart. Computer calculations were done for various values of $2h/a$, using 100 terms of the series. The results of these calculations are given in section 2.1.5 and are compared to the SGT and MSGT.

For the general n-wire cage, the image formulation becomes increasingly cumbersome because each charge gives rise to an image on each of the other cylinders, which gives rise to a second image on each of the other cylinders, etc. The location of the images becomes complicated in that they are not all located on a single radius from the wire center as they are for the two-wire case. For this reason, the image theory was not pursued for n larger than 2. If there was no other alternative, image theory would be tedious but useful. However, the method-of-moments approach described below has advanced to the point where it is easy to apply.

2.1.3 Conformal Transformation Theory

The field distortion from nearby wires for the two-wire case can also be solved by using the theory of conformal transformations (Poritsky in Clark, 1932). This also leads to an infinite series that must be truncated for calculations. This technique was used in the reference to evaluate the assumption that the field distortion caused by the proximity of the cylinders is the same as if the field in the vicinity of the cylinders is uniform. The conclusion was that the assumption gives results that are accurate when the wires are separated by more than 5 wire diameters. Conformal transformation and image theory results are numerically the same.

2.1.4 Method Of Moments

The method of moments (MOM) is a computer technique for solving a general set of linear equations (Harrington, 1968). In this case, the surface of the wires is divided into many segments, and the charge on each segment determined to match the boundary conditions. If enough segments are used, the solution can be considered to be exact as it accounts for field distortion due to the proximity of the wires to each other and to ground.

A commercially available computer code was used to calculate capacitance and electric field for the two-wire case (ELECTRO).[2] The results for the two-wire case are compared to the other methods of calculation in the following section. As an initial test, the MOM code was exercised for several cases of a single wire above ground and found to give numerically accurate results when the number of segments per wire was 16 or more.

2.1.5 Comparison Of Results (Two-Wire Case)

Figure 3a shows a comparison of calculated values of capacitance (C_2) for the two-wire cage with $2h/a$ = 5000. The results have been normalized by division by the capacitance of a single wire. Calculations were done using MOM, image theory, and SGT. Figure 3b gives the percentage error for the MOM and SGT by using the MOM calculations as the reference. Note that the MOM results are numerically the same as the image theory except when the wires are touching. The SGT results are within 1.5% when the wires are touching and better than 0.5% when $b/a > 2$.

2. ELECTRO-a computer program available from Integrated Engineering Software, Winnipeg, Manitoba, Canada.

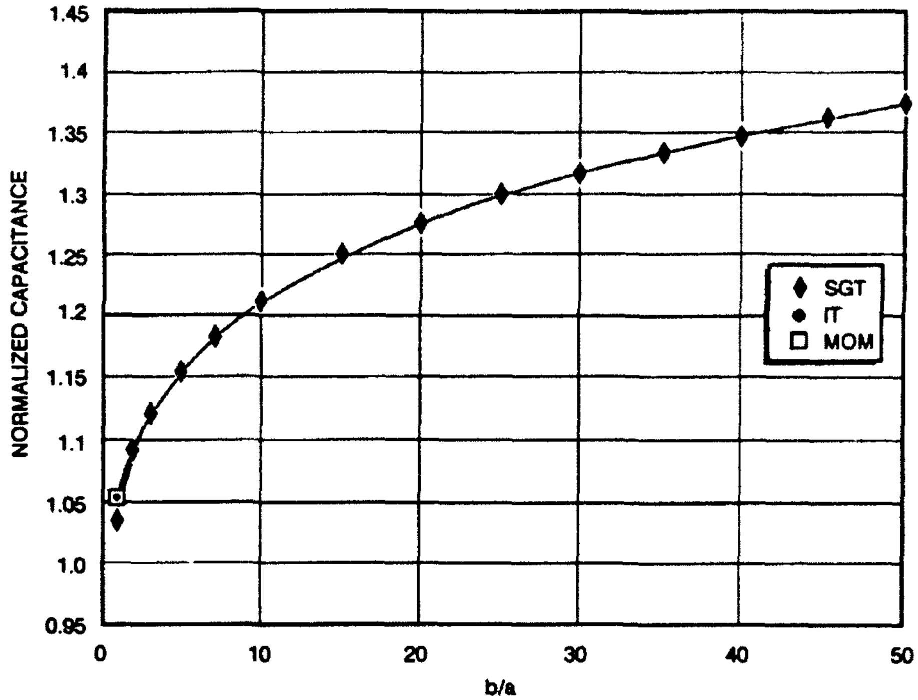

Figure 3a. Normalized capacitance—two-wire cage ($2h/a$ = 5000).

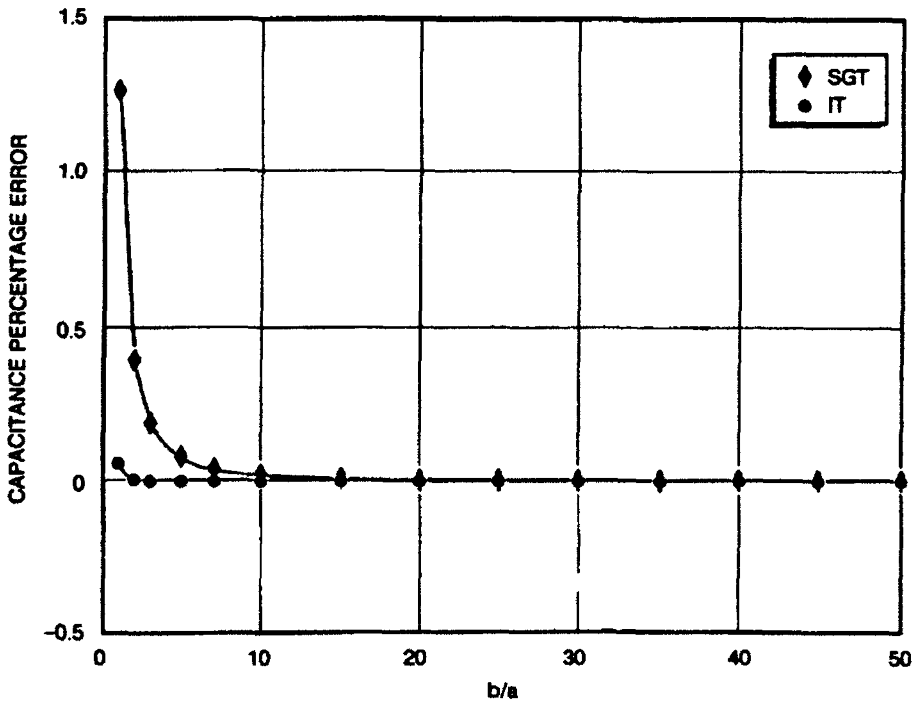

Figure 3b. Capacitance percentage error—two-wire cage ($2h/a$ = 5000) (MOM).

Figure 4a gives the maximum surface electric field for the same case. The calculations were done by using the same methods as above as well as CGT. Note that, for the two-wire case, the CGT formula is the same as the MSGT formula. Again, the results have been normalized to the value for a single wire. Figure 4b gives the percentage error for E-field, with MOM as the reference. Again, note that the MOM results are numerically identical to the image-theory results. Note that SGT does not give good results for small values of b/a. However, CGT gives excellent results for all values of b/a except when the wires are touching.

From the figures, it is concluded that MOM gives numerically exact results except when the wires touch. The MOM results will be used as the reference for cases involving three or more wires.

Another conclusion is that the SGT capacitance results are very accurate for all values of b/a. The maximum error, less than 2%, occurs when the wires touch.

For the two-wire case, the CGT gives excellent predictions for surface gradient.

Figures 5a and b, and 6a and b are similar to the above but with $2h/a = 300$. The SGT capacitance values are again very accurate for small values of b/a. However, in this case, they diverge at larger values. This is because the approximation that $D_{ij} = 2h$ is poor. Because of this assumption, the SGT capacitance formula is valid only when $h > 5b$.

2.2 GENERAL CASE, *n*-WIRE CAGE

2.2.1 MOM Results For 3, 4, 5, 6, and 12 Wires

The MOM has been validated in the above section for the geometry involved in multi-wire cages. The validation used the case of a two-wire cage and validated MOM against exact methods of calculation. When there are three or more wires, the exact image-theory calculations become very complicated. The only practical "exact" method of calculation is to use the method of moments.

To examine the accuracy of the simple formulas (SGT, MSGT, and CGT) for n greater than 2, a data set was developed by using MOM. The cases run were n = 3, 4, 5, 6, and 12, with $2h/a = 5000$. The results for capacitance and surface electric field are given in figures 7 and 8. The two-wire case has been included in the figures for completeness.

Note that, due to the geometry and the "exact" nature of the solution, the charge on each wire is not the same. The charge on the wires nearest to the image is slightly greater than the charge on wires farther away from the image. Thus, the electric field and capacitance is slightly different depending upon which wire is examined. For this reason, the plots in the figures include a maximum, a minimum, and the average value for both capacitance and surface electric field.

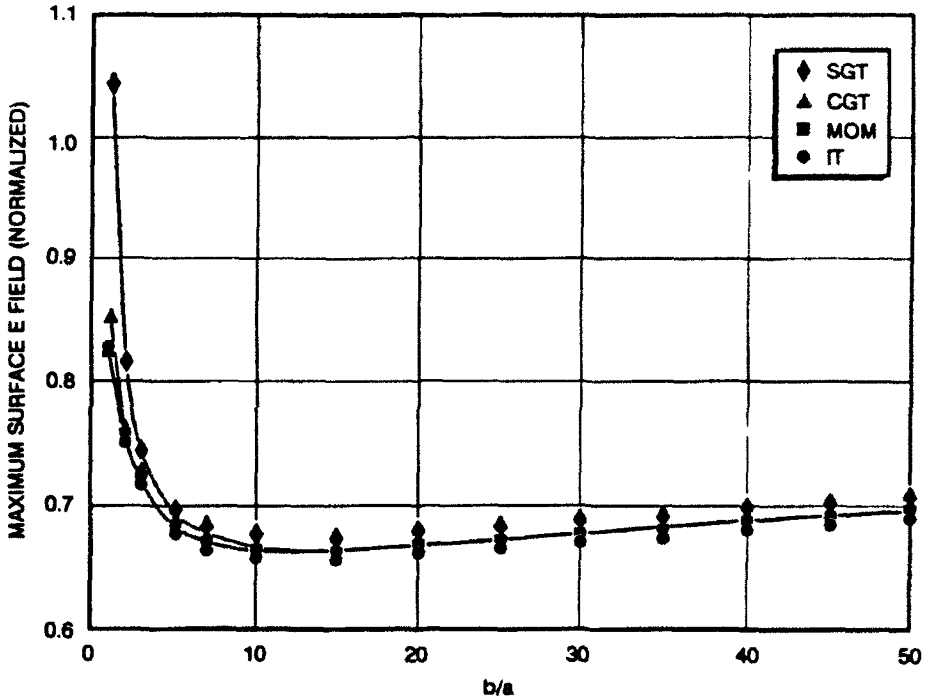

Figure 4a. Normalized surface E-field—two-wire cage ($2h/a$ = 5000).

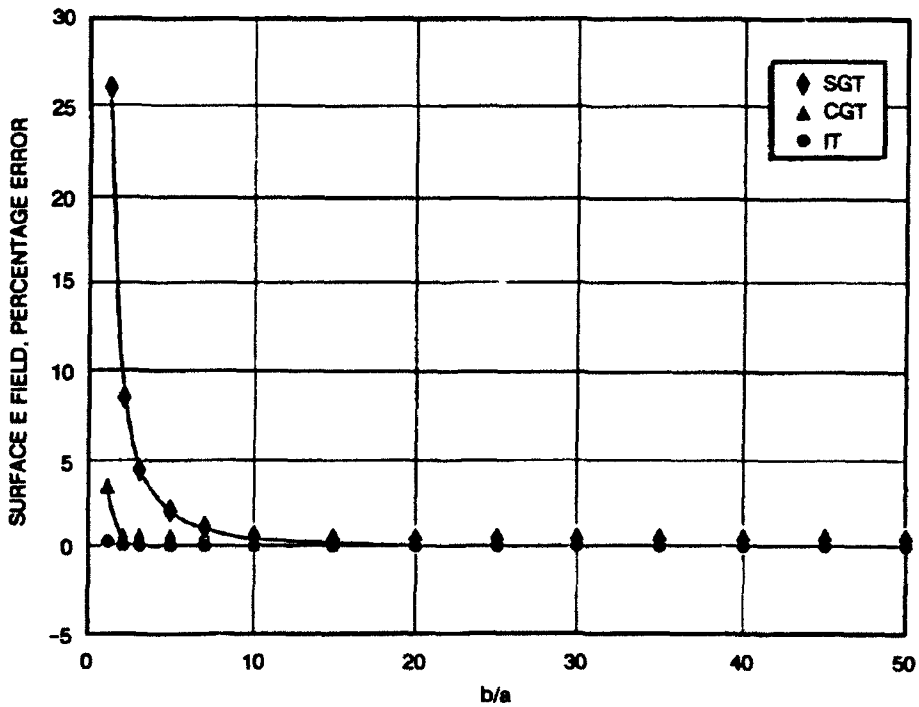

Figure 4b. E-field percentage error—two-wire cage ($2h/a$ = 5000) (MOM).

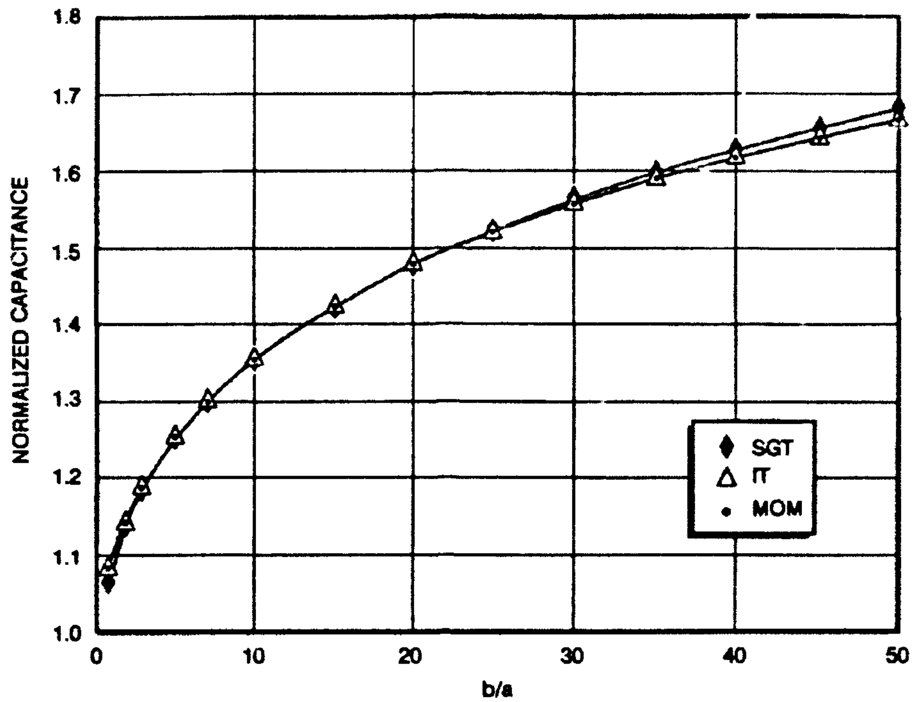

Figure 5a. Normalized capacitance—two-wire cage ($2h/a = 300$).

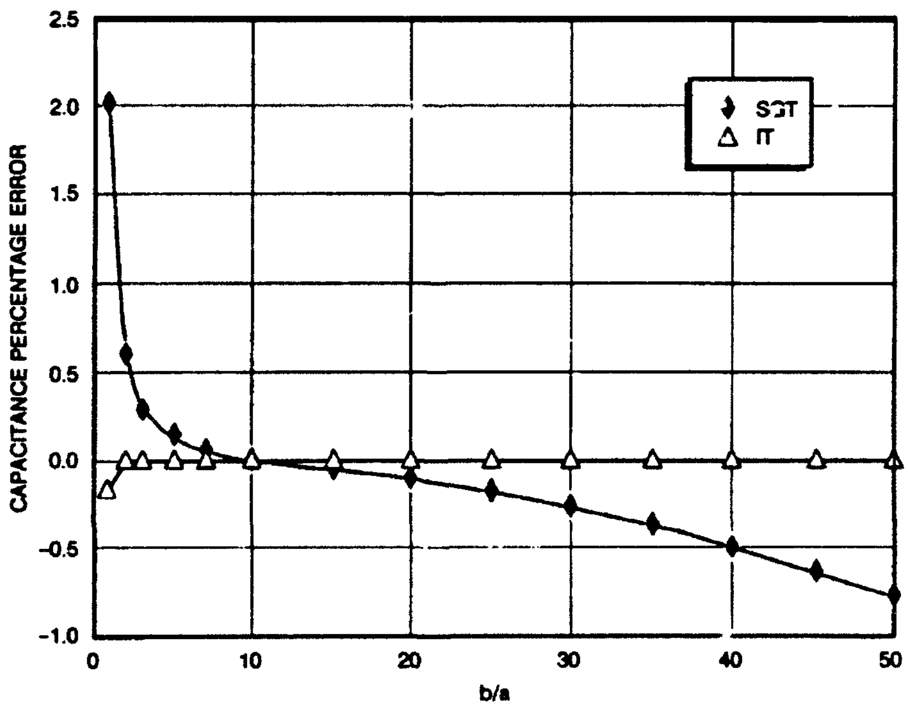

Figure 5b. Capacitance percentage error—two-wire cage ($2h/a = 300$) (MOM).

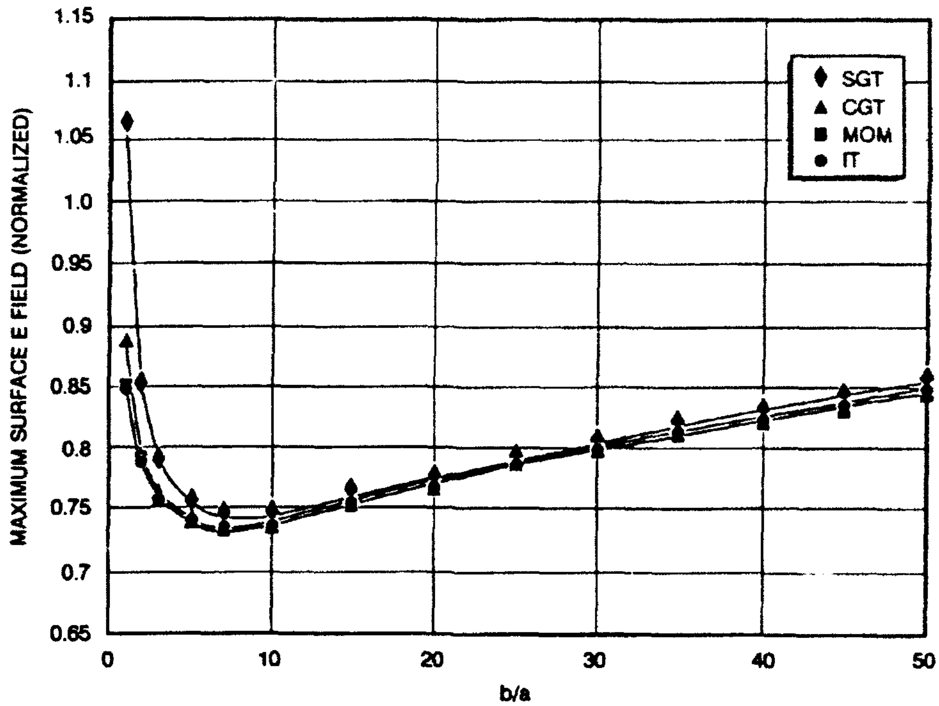

Figure 6a. Normalized surface E-field—two-wire cage ($2h$/a = 300).

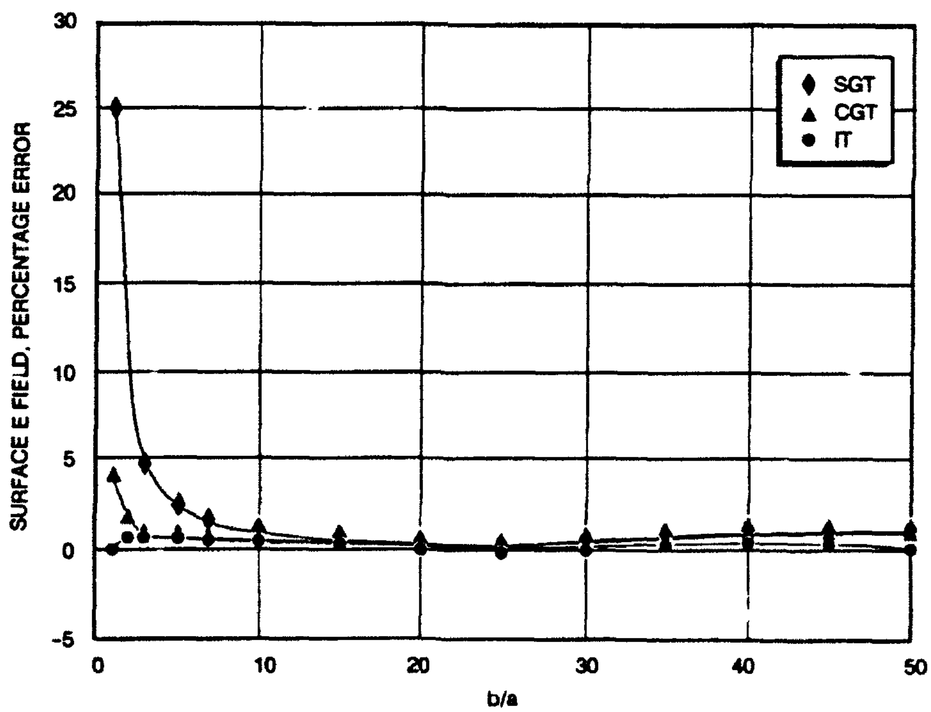

Figure 6b. E-field percentage error—two-wire cage ($2h/a$ = 300) (MOM).

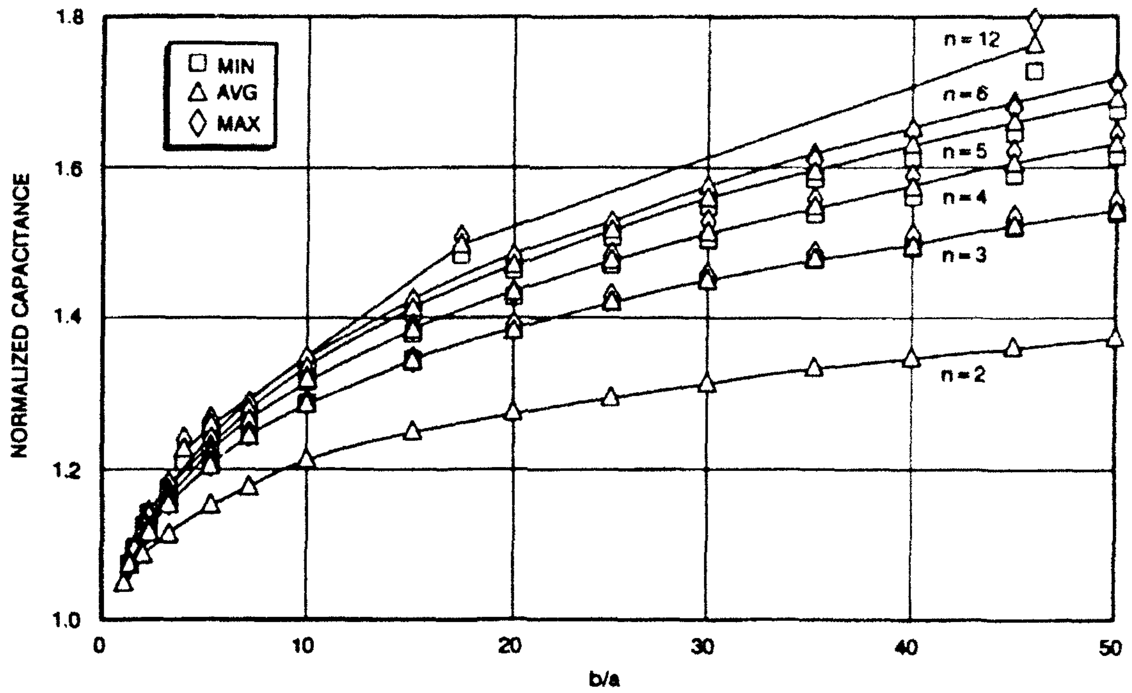

Figure 7. Normalized capacitance for several values of n, $2h/a = 5000$.

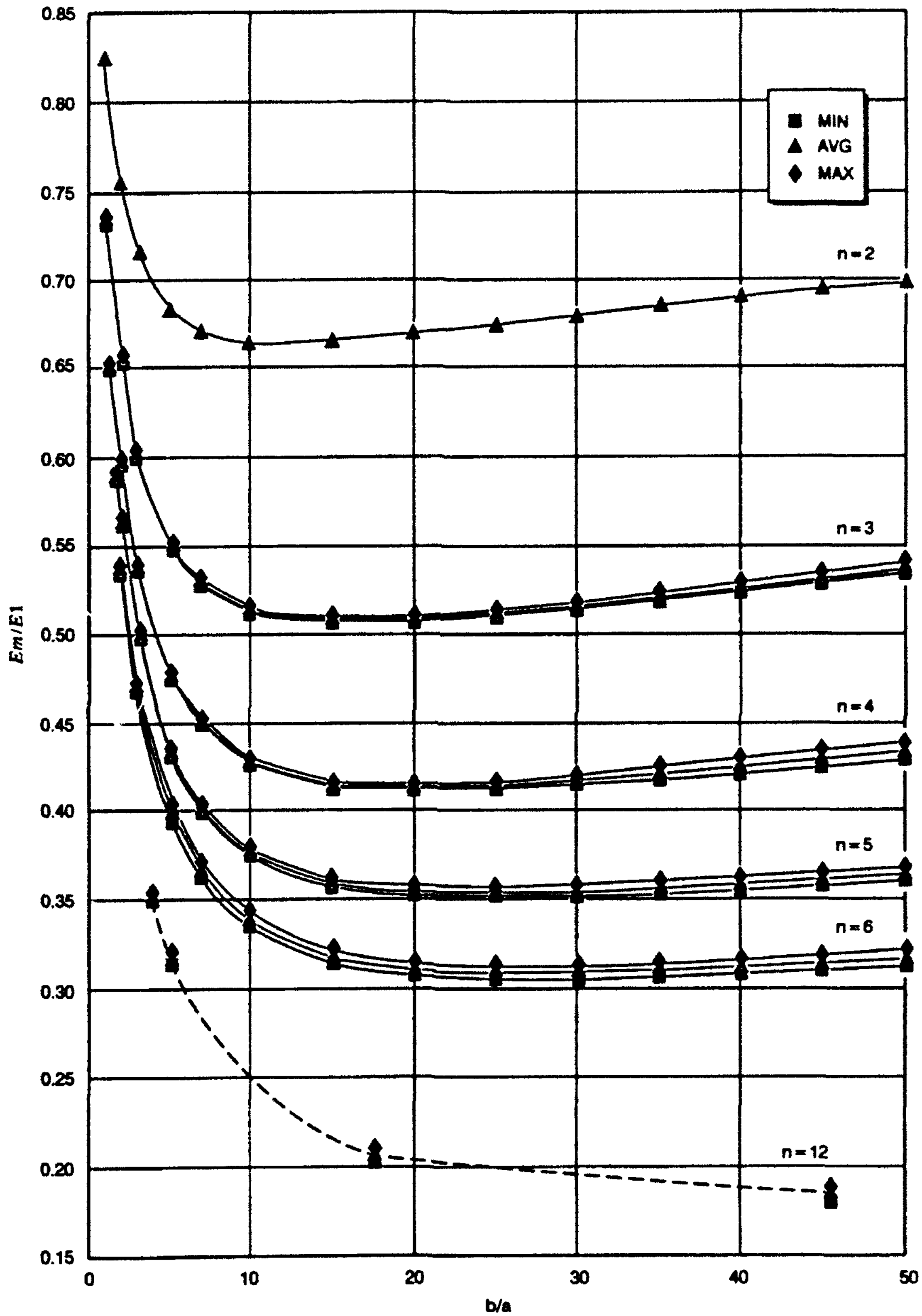

Figure 8. Maximum surface electric field (normalized) for several values of n, $2h/a = 5000$ (MOM).

2.2.2 Capacitance Comparison

The simple geometric theory for capacitance, as given in equation (3), has been compared with the value of capacitance calculated by the MOM. The results are presented in figures 9 through 13. The results are presented if the form of plots of the ratio of the SGT calculated value to the "actual" value are as calculated by using the MOM results. The comparison has been made to the maximum, minimum, and average values calculated by MOM.

These figures show that the value of capacitance, by using SGT, is always within 2.5% of the actual average value. For values of *b/a* greater than 3, the SGT gives values that are within 1% of the actual value.

The separation between maximum and minimum values increases with separation but in general is less than 1% for all values of *b/a* considered.

The conclusion is that the SGT capacitance formula is an excellent approximation!

2.2.3 Surface-Electric-Field Comparison

Three formulas for maximum surface electric field are to be examined, SGT, MSGT, and CGT. The comparison has been done by taking the ratio of the result calculated by the formulas, and the actual value as calculated by the MOM. The results are presented in figures 14 through 18. For the MSGT formula, the comparison has been made to the maximum, minimum, and average values calculated by MOM. The SGT and CGT have been compared only with the average value calculated by the MOM.

The SGT formula has large errors for small values of *b/a*, decreasing as *b/a* becomes larger. For example, for the three-wire cage with *b/a* of 1.5, the SGT error is in excess of 30%. Because of the large error for small spacings, the SGT formula gives an erroneous value for the separation having the minimum surface electric field.

CGT results have been calculated for $n = 3$ and 4. The error for the CGT method is significant at small spacings but less than for the SGT method.

The MSGT formula is best for all spacings. It has less than a 6% error at very small values of *b/a*, and less than 2% error for values of *b/a* of three or greater.

The MSGT formula gives the most accurate results of the simple formulas tried. It also gives the correct value of separation that has minimum surface electric field. This formula should be adopted as the standard formula for cage calculations.

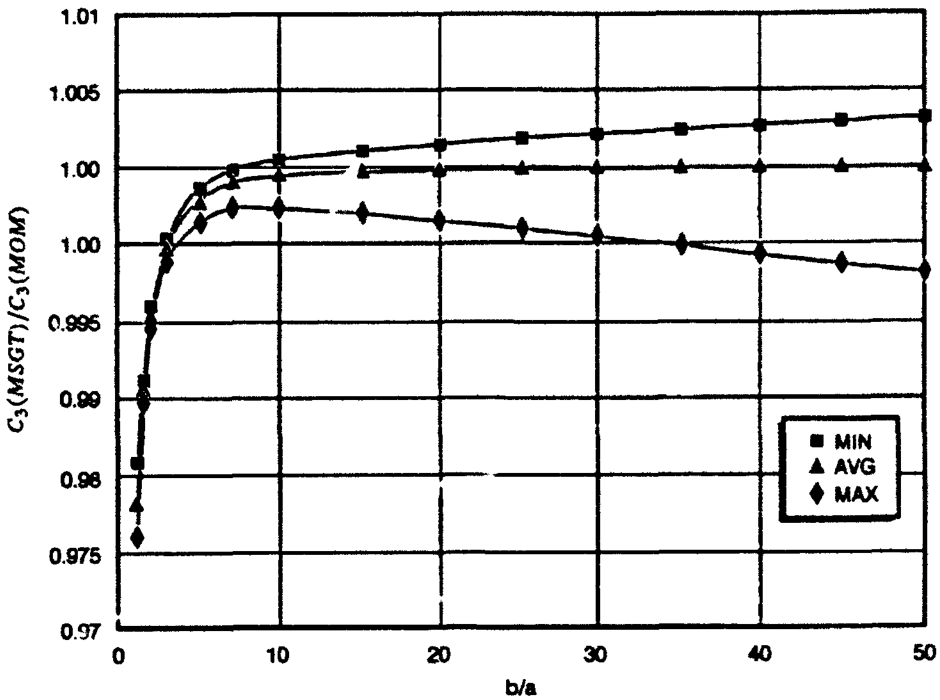

Figure 9. Capacitance comparison (MSGT/MOM)—three-wire cage ($2h/a$ = 5000).

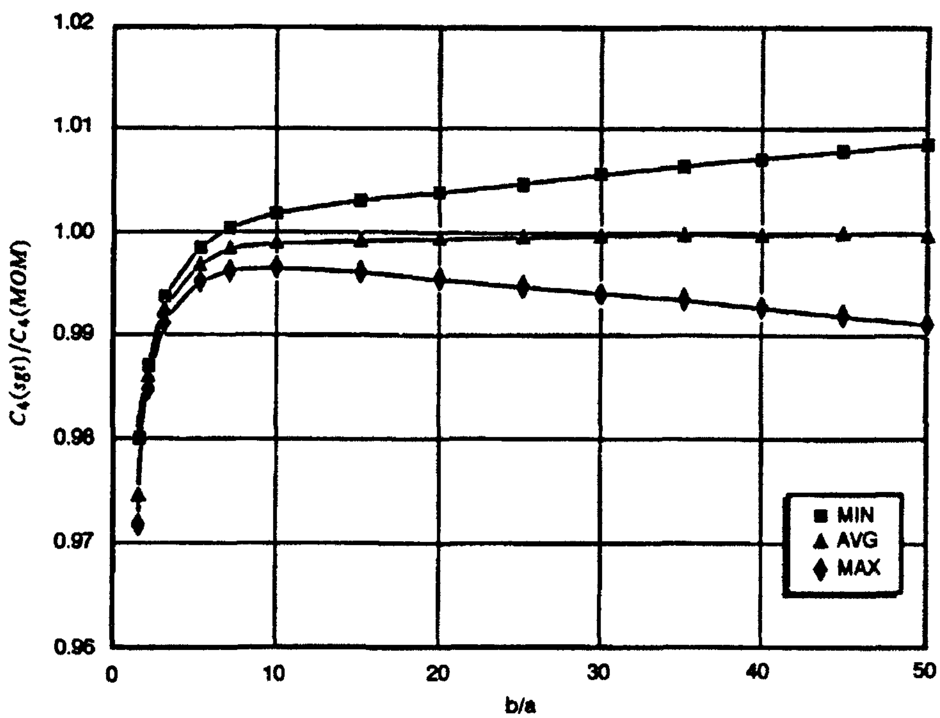

Figure 10. Capacitance comparison (SGT/MOM)—four-wire cage ($2h/a$ = 5000).

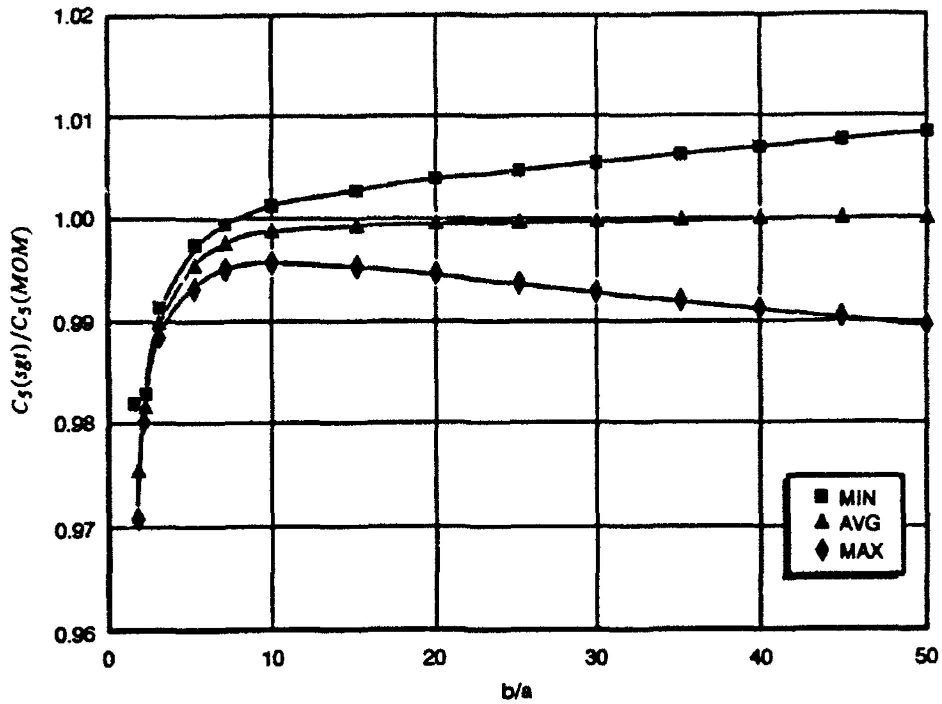

Figure 11. Capacitance comparison (SGT/MOM)—five-wire cage ($2h/a = 5000$).

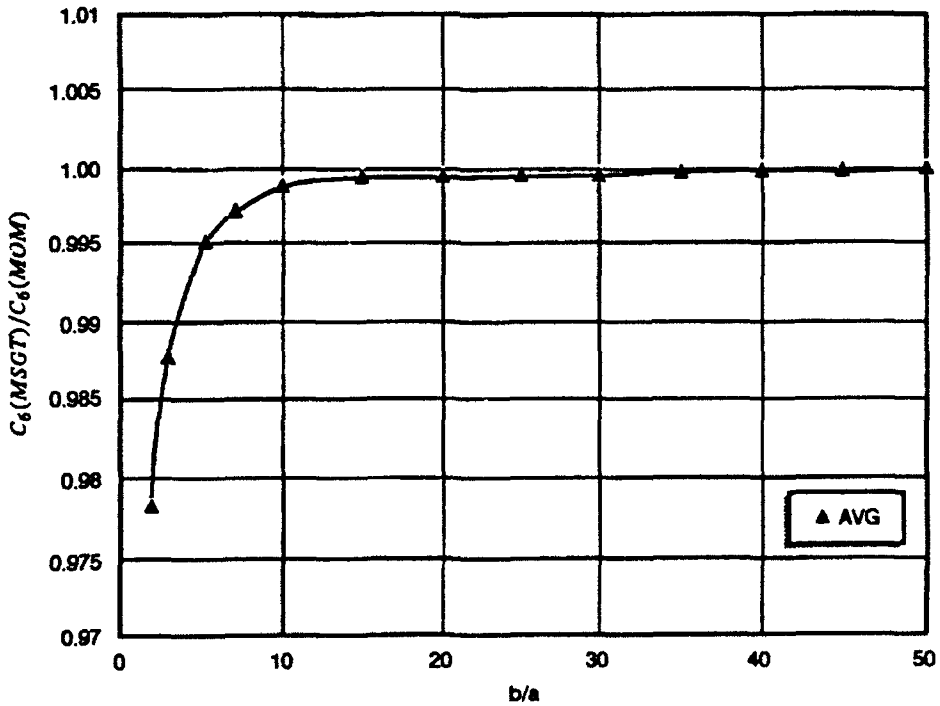

Figure 12. Capacitance comparison (SGT/MOM)—six-wire cage ($2h/a = 5000$).

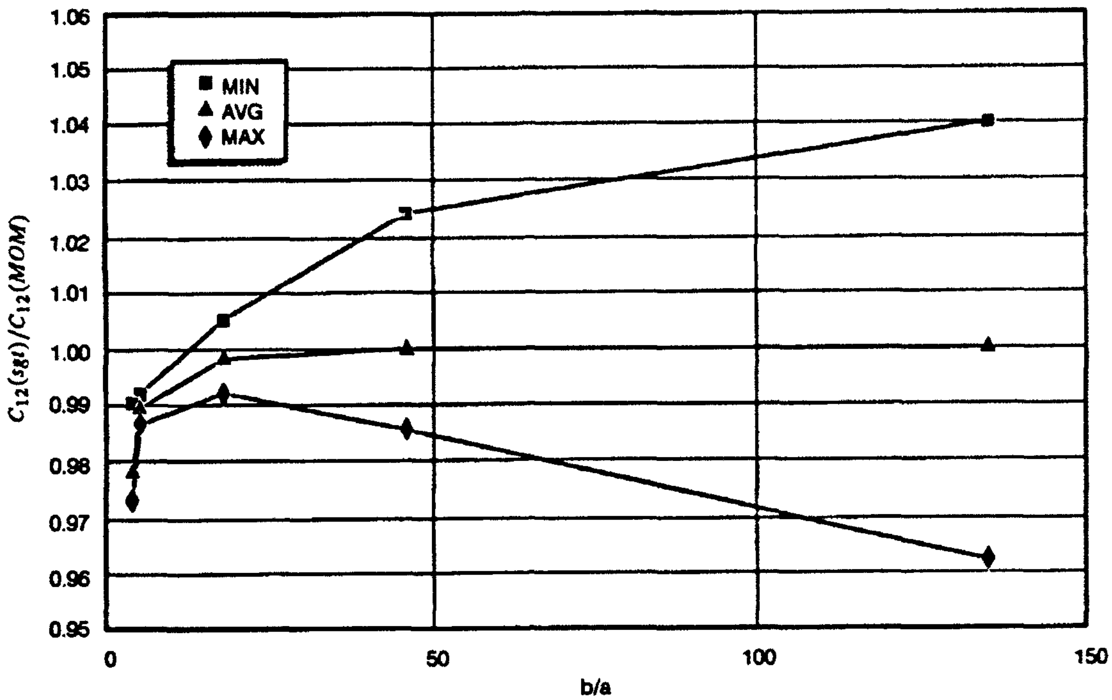

Figure 13. Capacitance comparison (SGT/MOM)—twelve-wire cage ($2h/a$ = 5000).

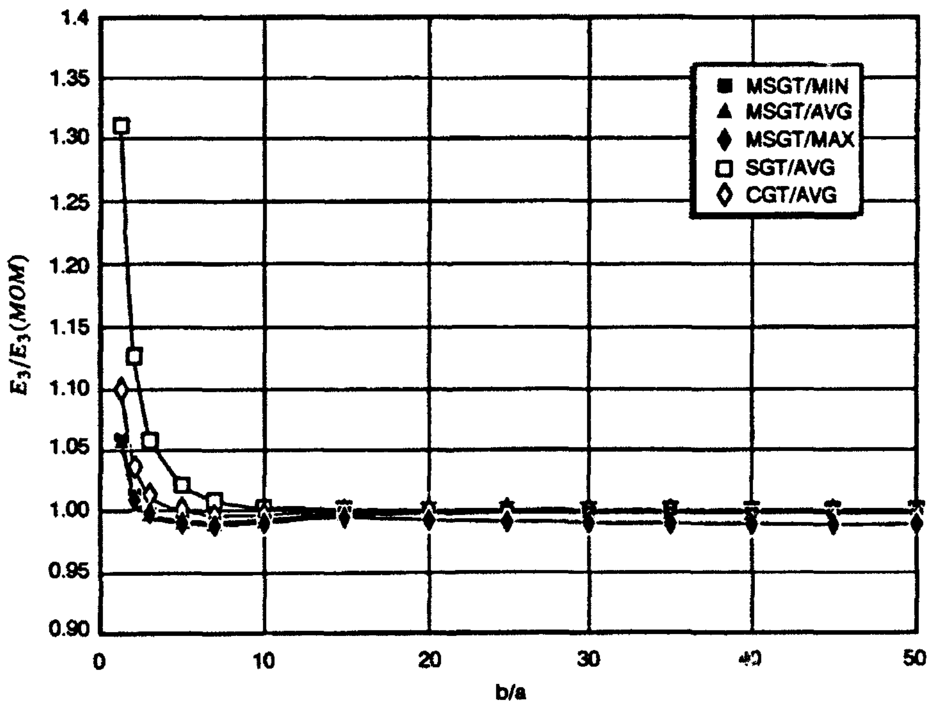

Figure 14. Normalized surface E-field comparison to MOM—three-wire cage ($2h$/a = 5000).

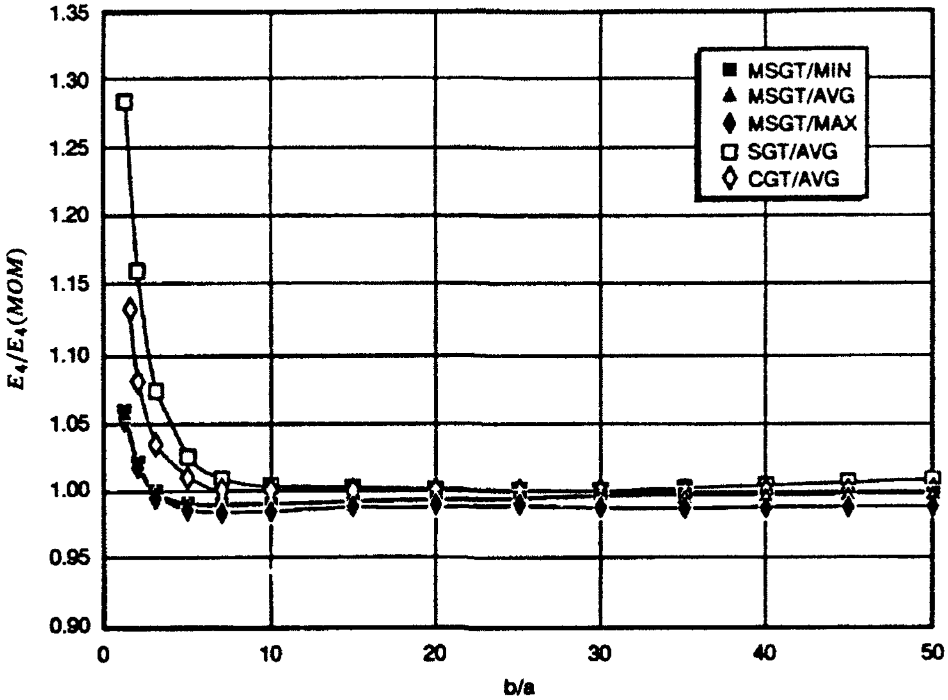

Figure 15. Maximum surface E-field comparison to MOM—four-wire cage ($2h/a = 5000$).

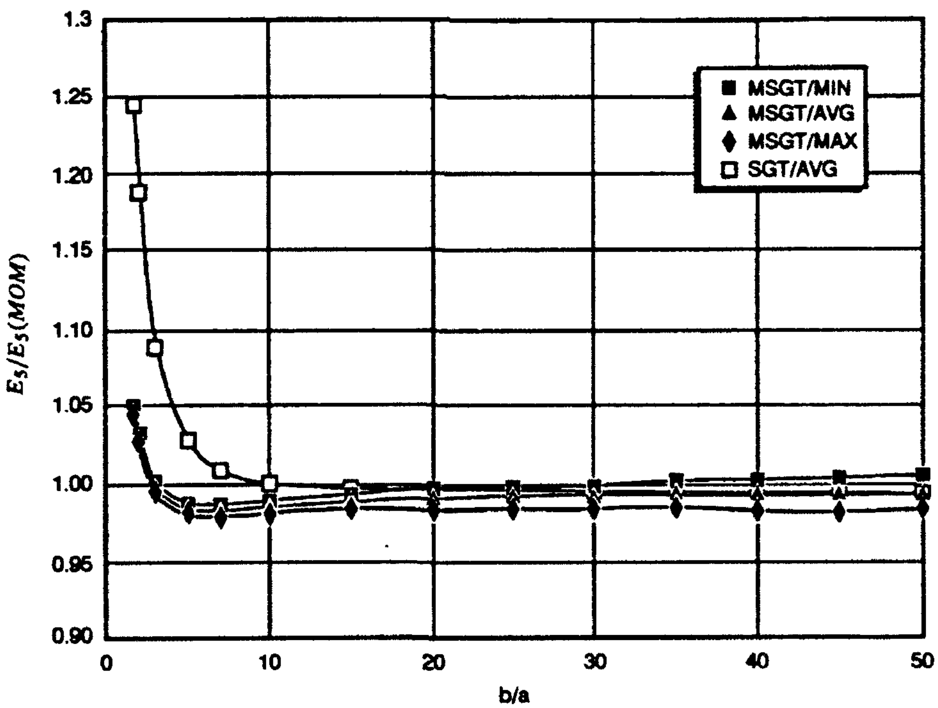

Figure 16. Maximum surface E-field comparison to MOM—five-wire cage ($2h/a = 5000$).

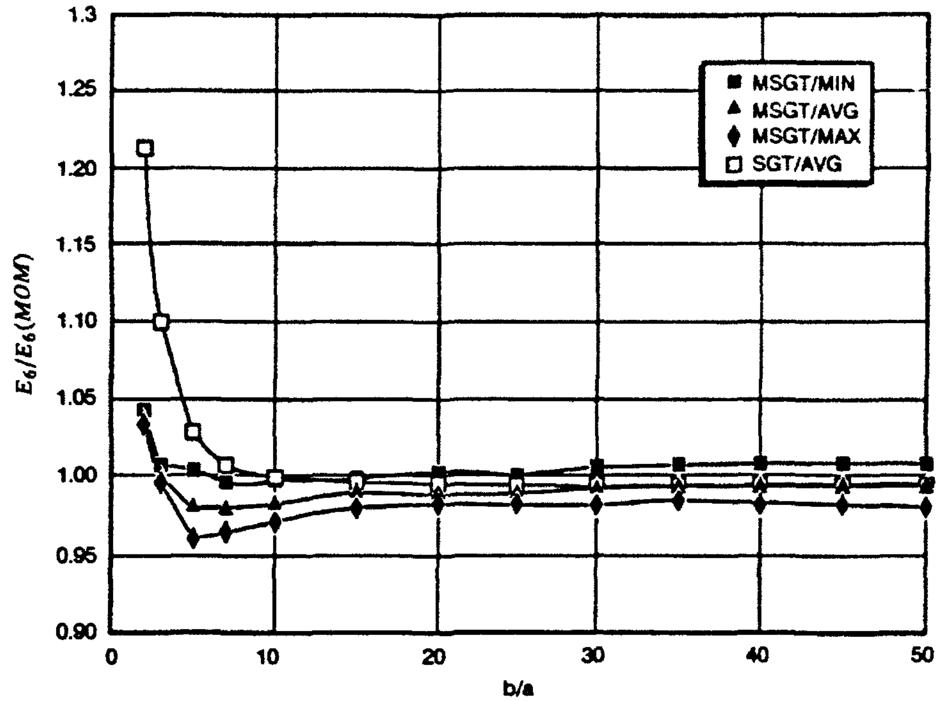

Figure 17. Maximum surface E-field comparison to MOM—six-wire cage ($2h/a = 5000$).

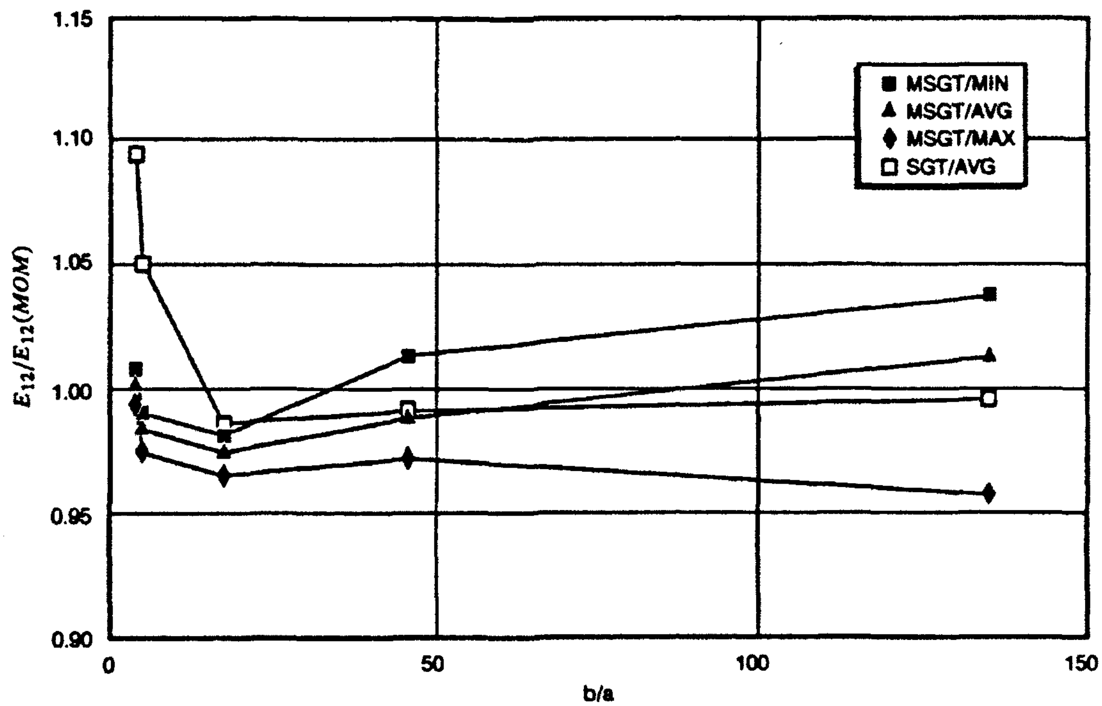

Figure 18. Maximum surface E-field comparison to MOM—twelve-wire cage ($2h/a = 5000$).

3.0 DESIGN

3.1 DESIGN DATA

Design curves have been developed by using the MSGT formula.

Figure 8 shows that, at first, the surface gradient decreases rapidly with increasing *b*, then, more slowly, goes through a broad minimum and then slowly increases.

The reason for the minimum is as follows: The surface gradient is the product of a capacitance factor and a geometry factor. The capacitance factor increases with separation while the geometry factor decreases with separation. When wires are less than one wire radius apart, the product is somewhat less than the value for a single isolated wire. At large separations, both factors and the product are asymptotic to the value for a single isolated wire. For close separations, the geometry factor falls off much faster than the capacitance increases and, thus, the product of the two factors has a minimum value.

The capacitance factor is a logarithmic function of height. Thus, the rate increase of the capacitance is reduced as height increases. The rate decrease for the geometric factor is independent of height. Thus, the location of the minimum is a function of height. The minimum occurs at wider spacings as height increases.

A computer program based on MSGT was developed to find the value of b/a, for which the surface gradient is minimum. This program is included in appendix F. The location of the minimum is called b_{min}/a and is plotted versus $2h/a$ in figure 19. From figure 19, b_{min}/a is seen to increase approximately logarithmically with height, as well as linearly with n.

The wire-to-wire spacing at minimum gradient is called S_{min}, and is plotted in figure 20. From this figure, it is seen that S_{min} is nearly the same for $n = 3$ to 6. S_{min} also varies logarithmically with height. Curiously, the two-wire case has a different slope than all the other cases. As n increases, the slope of S_{min} remains nearly the same as cases $n = 3$, but the curve moves down slightly with n. This is illustrated in figure 20 by the curve for $n = 12$.

Table 1 gives values of S_{min} in terms of wire diameters for practical heights. For 1-inch-diameter wires, the corresponding heights in the table are 20.8, 208, and 2083 feet. Table 1 shows that for minimum gradient, wire-to-wire spacings need not be more than 8- to 15-wire diameters for the two-wire case and slightly more when n is greater than 2.

Table 1. $S_{min}/2a$ for practical values of $2h/a$.

n	$2h/a$ 1000	$2h/a$ 10,000	$2h/a$ 100,000
2	8.86	13.11	17.45
3–6	9.75	15.5	21.5

3.1.1 Equivalent Radius for Electric Field

The concept of equivalent radius for electric field (a_{eqE}) of a cage is a useful design concept that was introduced by Miller (1956). This concept is defined as the radius of a

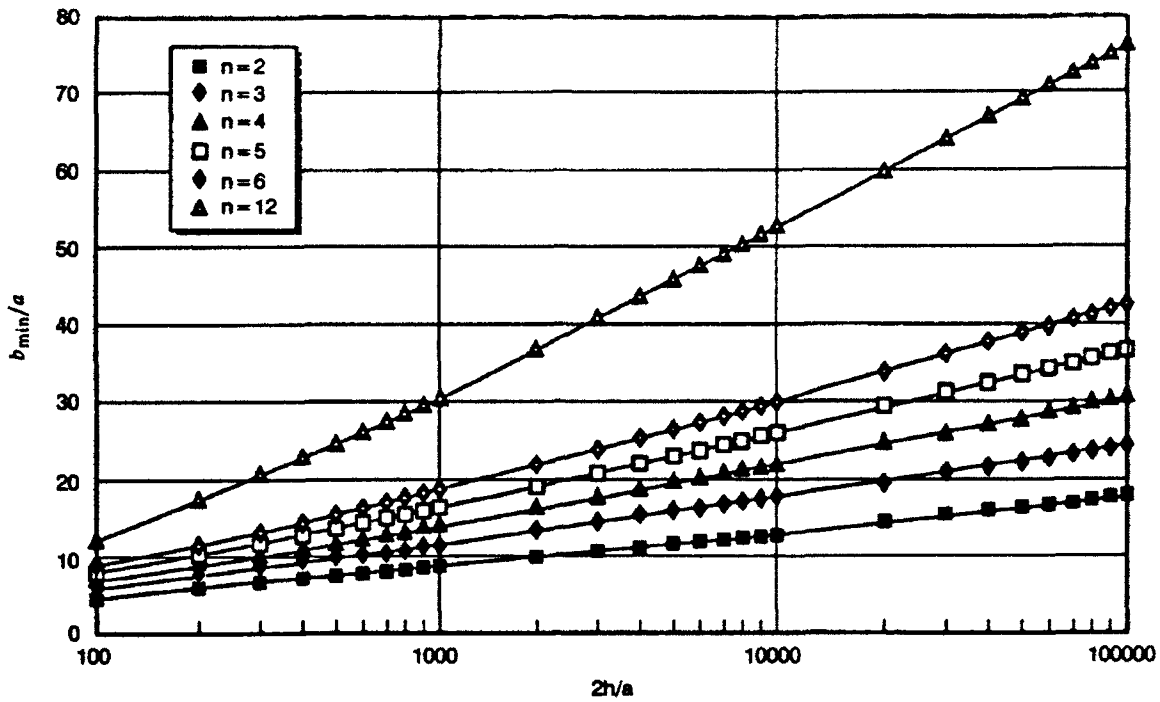

Figure 19. n-wire cage— b_{min}/a versus height.

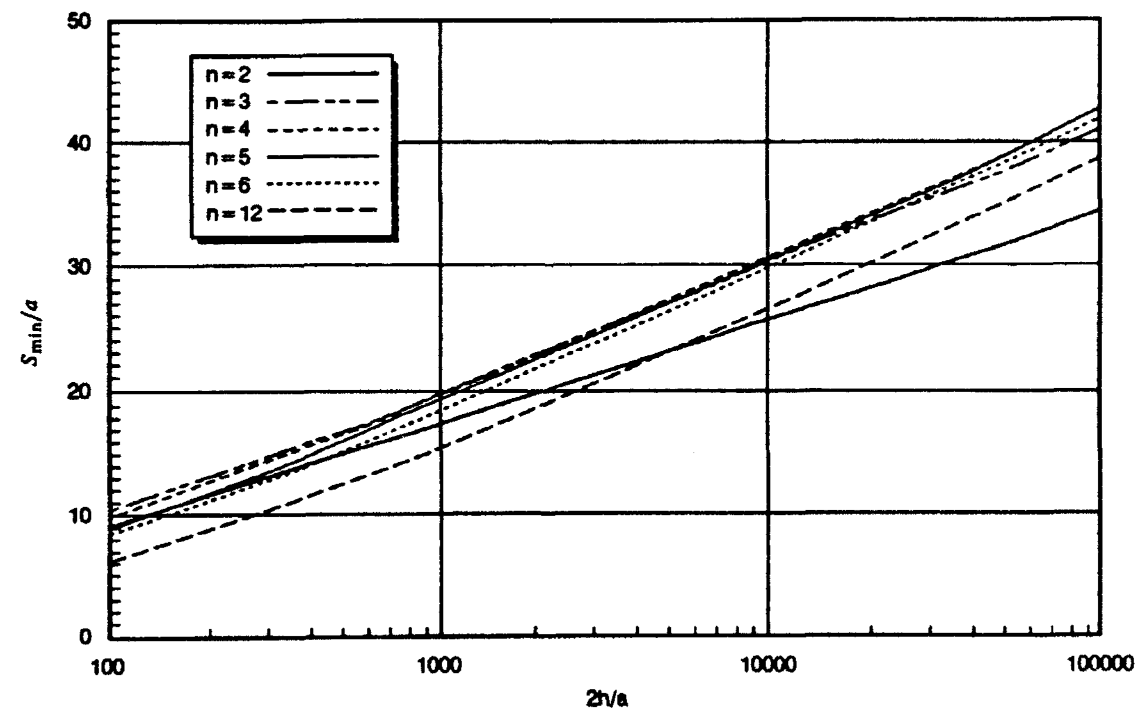

Figure 20. n-wire cage—S_{min}/a versus height.

single wire that would have the same maximum surface gradient as the cage when located at the same position as the cage axis.

It is defined by the equation

$$E_m = \frac{V}{a_{eqE}}\left[\frac{1}{\ln\left(\frac{2h}{a_{eqE}}\right)}\right].$$

Note that this equation has the same form as equation (2).

Using normalized parameters gives the following equation for a_{eqE}/a ,

$$\frac{E_m}{E_1} = \frac{a}{a_{eqE}}\left[\frac{\ln\left(\frac{2h}{a}\right)}{\ln\left(\frac{2h}{a}\,\frac{a}{a_{eqE}}\right)}\right].$$

This transcendental equation can be solved by iterative search or plotting. An iterative search is included in the computer program that finds b_{min} to give the corresponding value of a_{eqE}/a. A sample of the output of this program is included in appendix F. The values of a_{eqE} for $b = b_{min}$ are useful for design and are plotted in figure 21.

It is interesting that the abscissa of figure 21 can be converted from $2h/a$ to $2h/a_{eqE}$ by dividing the abscissa values by the value of the function (a_{eqE}/a). This gives a_{eqE}/a as a function of $2h/a_{eqE}$ for cages with radius $b = b_{min}$. A set of these curves for values of n from 2 through 6 and $n = 12$ are given in figure 22. These curves are even more useful for design.

Finally, it was observed that the curves of figure 19 could be normalized to a_{eqE} by dividing the values of both the ordinate and abscissa by the value of the function a_{eqE}/a at each point. The resulting set of curves, given in figure 23, completes the design data. Once a value for a_{eqE} has been determined, the curves of figures 22 and 23 can be used to determine the optimum cage dimensions meeting the surface electric field requirement.

Note that the curves of figure 23 are nearly straight lines on the linear-log scale. All the curves have nearly the same slope. These curves can be approximated by the following simple formula:

$$b_{min} = 2.36 \times \mathrm{Log}_{10}[a_{eqE}/a] - I_n ,$$

where I_n is the intercept value and is given in table 2.

Table 2. Intercept values for b_{min} equation.

n	I_n
2	0.8939
3	0.7184
4	0.6095
5	0.5216
6	0.4684
12	0.2982

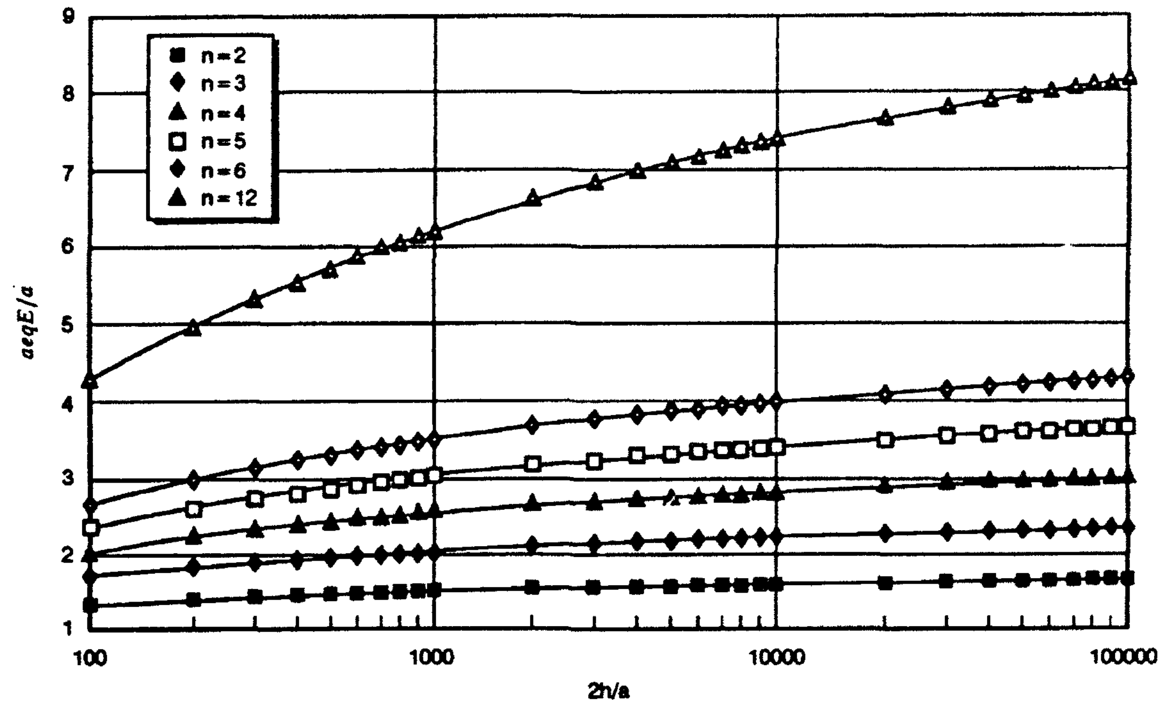

Figure 21. Two-wire cage—a_{eqE}/a versus height.

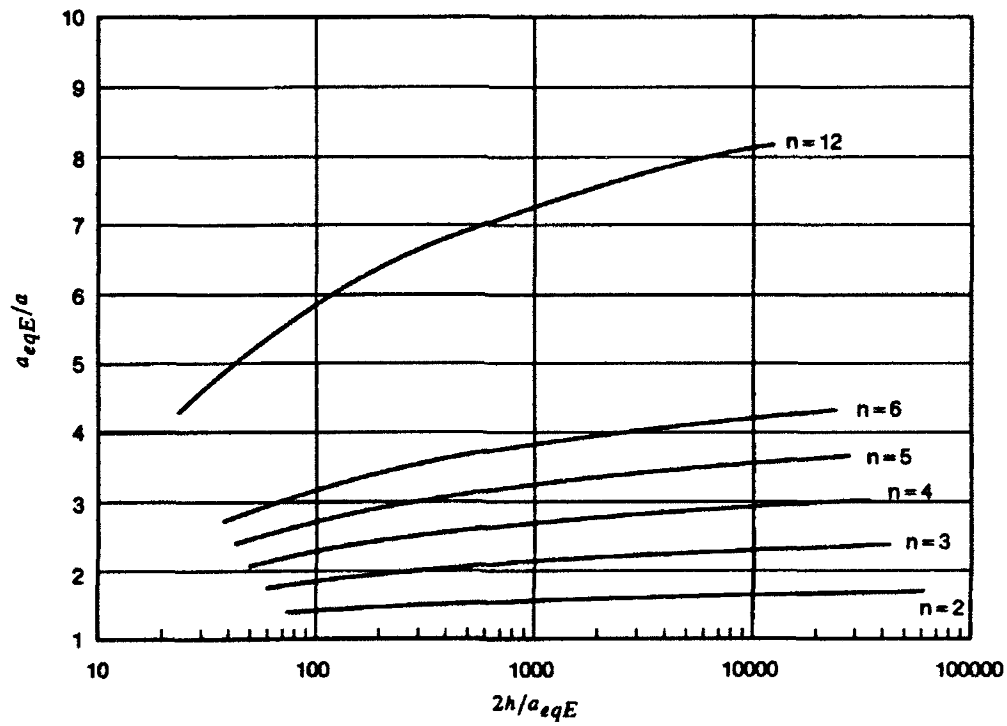

Figure 22. a_{eqE}/a versus $2h/a_{eqE}$ for various values of n.

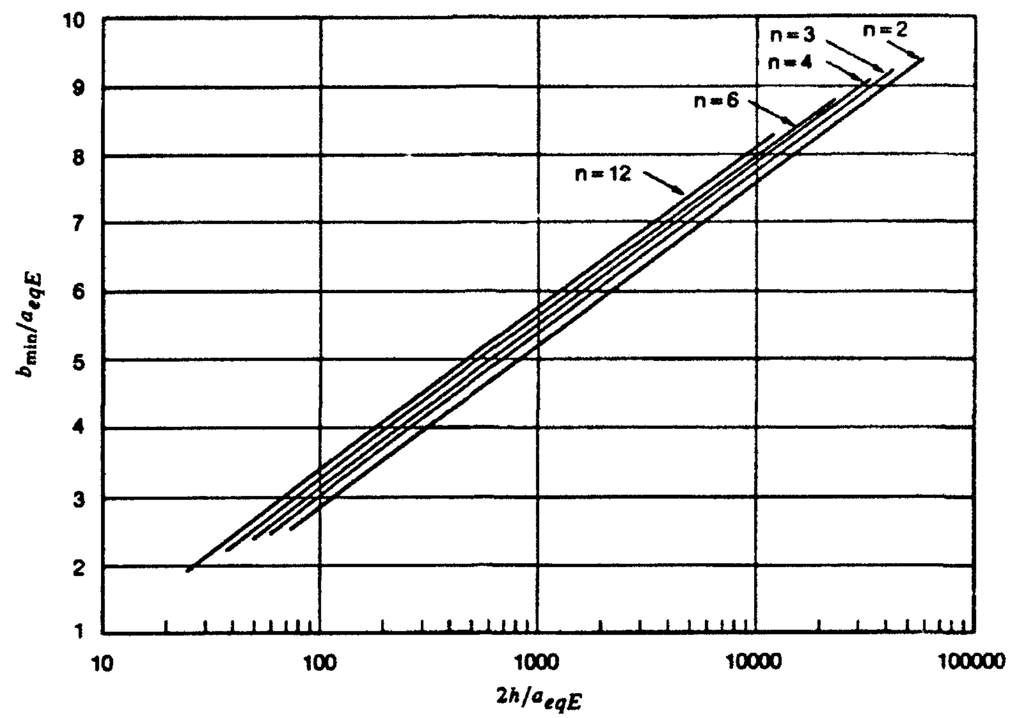

Figure 23. b_{min}/a_{eqE} versus $2h/a_{eqE}$ for various values of n.

3.2 DESIGN PROCEDURES

A major design consideration for cages is to minimize the structural impact on the rest of the structure. Increasing the spacing of the cage increases the projected wind area, and the spreader hardware weight. Cost considerations and wire availability often dictate the size of wires available to the designer. For these reasons, the first design procedure starts with a given diameter of wire to be used in the cage and determines the minimum spacing that will have a satisfactory gradient.

3.2.1 Minimum Spacing

For this case, the following parameters are given:

height h
wire radius a
design voltage V
design gradient E_0

The design procedure consists of finding the value of the cage radius b that has a gradient equal to E_0. The approach is to find E_0/E_1 using the single wire formula, and then plot equation (5) for several values of n. The design value of b is selected as the value where $E_n/E_1 = E_0/E_1$. Safety factors are included in the selection of E_0. To minimize the structural loading effects, the minimum value of n is chosen that satisfies the gradient criteria.

A. First Design Example (Given Wire Size)

As an example, select 1-inch-diameter wires, 400 feet above the ground, operated at 200 kV with a design gradient of 1 kV/mm. The value of $2h/a$ = 19,200. The design parameter is

$E_0/E_1 = 0.6263 = E_n/E_1$.

Figure 24 has plots of E_n/E_1 for several values of n.

Note that the $n = 1$ curve is a special case for the single wire. It is the ratio of E for a wire with radius b, to E for a 1-inch-diameter wire. This curve can be used to find a_{eqE}, which is the ordinate value where $E_n/E_1 = 0.6263$.

From figure 24, the following conclusions can be drawn:

(1) The $n = 1$ case gives $a_{eqE}/a = 1.68$.

(2) For 1-inch-diameter wires, there is no spacing that will give an acceptable gradient for $n = 2$.

(3) For $n = 3$, the design criteria will be met when $b/a = 2.4$.

(4) For $n = 4$, the cage radius is only slightly more than that of a single wire meeting the criteria. For higher values of n, the minimum radius of the cage will be larger than the radius of a single wire that meets the criteria.

Therefore, the best cage meeting the criteria is a three-wire cage with a cage radius of 1.2 inches using 1-inch wires.

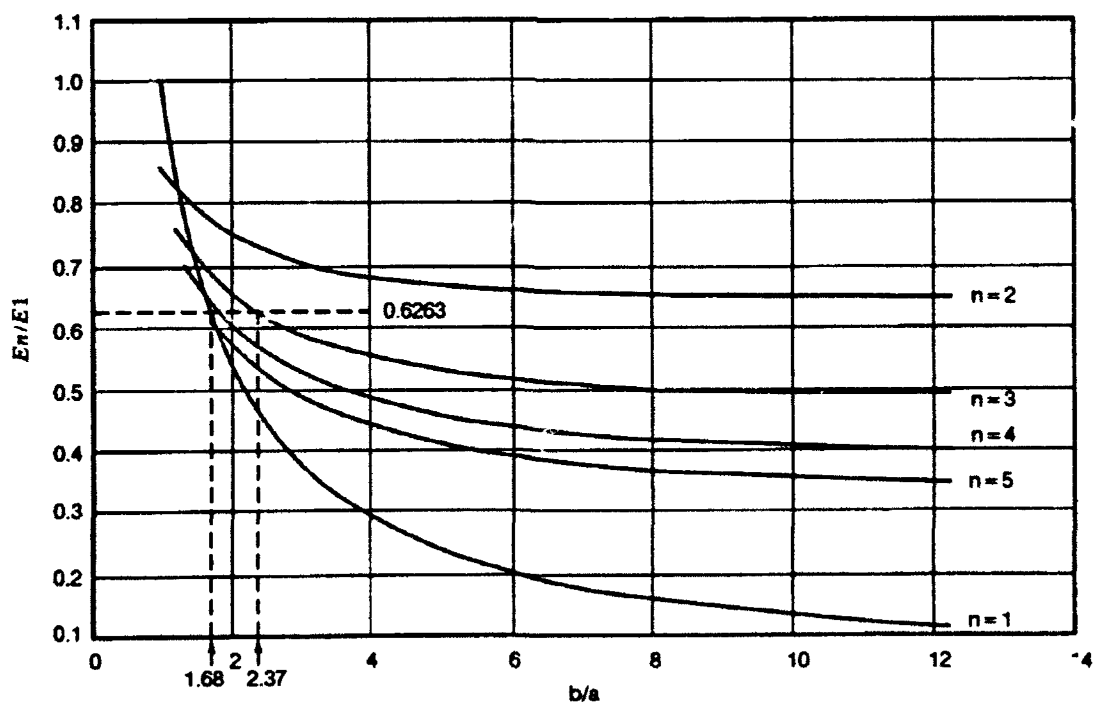

Figure 24. First design example, E_n/E_1 for various values of n ($2h/a$ = 19,200).

3.2.2 Minimum Wire Cross Section

For the second design procedure, the wire diameter is not specified but will be determined. The design parameters are the same as above, but wire radius is not specified. This design procedure results in a cage with the minimum cross sectional area that meets the gradient design criteria. It starts with determination of the required radius for the single wire that will meet the design criteria. This is the required a_{eqE}.

Once a_{eqE} is known, calculate the parameter $2h/a_{eqE}$, and then a_{eqE}/a is determined by figure 22 for each value of n chosen. The required value of wire radius for the cage is determined by

$$a = \frac{a_{eqE}}{\left(\dfrac{a_{eqE}}{a}\right)} .$$

Next, determine b_{min}/a from figure 23 for each value of n.

The above procedure defines the cage parameters b and a. As a check, use these parameters in equation (5) to be sure the design criteria is met.

A. First Design Example (Minimum Wire Cross Section)

This example uses the same design parameters as in section 3.2.1.

Design Parameters
Height h = 400 feet
Voltage V = 200 kV
E_0 = 1kV/mm

From the above example, we already know that the required radius of a single wire is

a_{eqE} = 1.68 × 0.5 inch = 0.84 inch.

This gives a value of $2h/a_{eqE}$ = 11,429. From the curves in figures 22 and 23, determine a_{eqE}/a and b_{min}/a. The design procedure is demonstrated by table 3. The results are progressively obtained moving from left to right in the table. The notation at the bottom of each column indicates how the results in that column were derived. The following conclusions follow from table 3.

(1) The minimum required wire diameter for a two-wire cage is 2 × 0.515 = 1.03 inches. This is consistent with the result of the first part of this example that indicated that there was no two-wire solution for 1-inch-diameter wires.

(2) The minimum cross-section cage radius is about the same for all values of n.

(3) An optimum solution can be found for any value of n. The procedure outlined finds these solutions. The E_n column is the calculated value of maximum gradient using equation (5). Note that the design criteria is met exactly!

(4) The area column contains the cross-sectional area of the cage wires. The weight and strength of the cage will be proportional to this value. The total cross-sectional area of the cage wires decreases significantly as n increases.

Consequently, the weight of the cage will be less for larger values of n. The weight will not decrease as much as the cross-sectional area because of the added weight of the spreader-hardware. The cross-sectional area decreases by a factor of 4.12 from 1 wire to 12 wires.

(5) The windage column contains the projected profile for windage. The profile is calculated by assuming each wire acts independently and uses a factor of 1.2 to account for the cylindrical shape of the wires.[3] This windage profile increases with n, indicating that wind loading will be higher if more wires are used. The windage increases 48% from 1 wire to 12 wires.

Table 3. First design example–minimum wire cross-section wire diameter and cage radius for various values of n, 200 kV, h = 400 feet, $a_{eqE} = 0.84$inch, $2h/a_{eqE} = 11,429$, E_0 = 1 kV/mm.

n	a_{eqE}/a	b_{min}/a_{eqE}	a inch	d inch	b inch	a_{eqC}	E_n	Area inch2	Windage inch
1	1		0.84	1.680			1.00	2.217	2.016
2	1.63	8.75	0.515	1.031	7.350	2.752355	1.00	1.669	2.474
3	2.27	8.93	0.370	0.740	7.501	3.96776	1.00	1.291	2.664
4	2.932	9	0.286	0.573	7.560	4.71721	1.00	1.031	2.750
5	3.559	9.063	0.236	0.472	7.613	5.243548	1.00	0.875	2.832
6	4.2	9.125	0.200	0.400	7.665	5.627174	1.00	0.754	2.880
12	8.1	9.25	0.104	0.207	7.770	6.670124	1.00	0.405	2.987
	Fig 22	Fig 23	0.84/col2	2*a	0.84/col3		Eqn 5	n*Pi*a^2	1.2*n*a

3.3 SECOND DESIGN EXAMPLE

This example corresponds to the end of span E-5 in the east array at VLF Jim Creek. The end of this span is quite far from the other spans and is treated as if it were a single isolated wire.

The existing span is a copper weld wire 1.01 inches in diameter. The end of the span is 600 feet away from ground. Given voltages of 200 kV and 330 kV and a design gradient of 1 kV/mm design an adequate cage for this wire.

3.3.1 Given Wire Size

First, calculate the gradient on the wire for the two design voltages. The simple single-wire formula can be used because the end effect is assumed to be mitigated by the corona rings on the insulators.

a = 1.01 inches
h = 600 feet
E_0 = 1.00 kV/mm

3. Personal communication with Bob Prince, NAVFAC Chief Engineer for Towers.

$2h/a = 28{,}515$
$V = 200$ kV $\quad E_1 = 1.52$ kV/mm $\quad E_0/E_1 = 0.66$
$V = 330$ kV $\quad E_1 = 2.50$ kV/mm $\quad E_0/E_1 = 0.40$

Figure 25 has the corresponding plot of equation (5). The value of a_{eqE} and b/a versus n are obtained from the figure as the points where E_n is equal to 0.66 and 0.40 respectively. The results are given in tables 4a and 4b. The positions with an X indicate either no solution or a single wire will have the same or a smaller total diameter as the cage.

The following conclusions can be drawn from tables 4a and 4b.

(1) For a fixed wire size, both weight and windage increase with n. Therefore, the cage with the smallest value of n that satisfies the design criteria is selected.

(2) For 200 kV, a two-wire cage with 5-1/2 (2 × 2.75) -inch separation would be adequate.

(3) For 330 kV, a four-wire cage would be required.

Table 4a. Second design example—given wire diameter and cage radius for various values of n, a = 0.505, $2h/a$ = 28,515 200 kV, a_{eqE} = 0.803 inch.

n	a_{eqC}/a	b/a	a inch	d inch	b inch	E_n	Area	Windage
1		1.590	0.803	1.606		1.00	2.025	1.927
2	1.664203	5.430	0.505	1.010	2.74215	1.00	1.602	2.424
3	1.125121	1.920	0.505	1.010	0.9696	1.00	2.404	3.636
4	0.987295	1.540	0.505	1.010	0.7777	1.00	3.205	4.848
5	X							
6	X							

Table 4b. Second design example—330 kV, a_{eqE} = 1.49 inches.

n	a_{eqC}/a	b/a	a inch	d inch	b inch	E_n	Area	Windage
1		2.790	1.409	2.818		1.00	6.237	3.381
2	X							
3	X							
4	4.198488	10.610	0.505	1.010	5.35805	1.00	3.205	4.848
5	2.713159	5.470	0.505	1.010	2.76235	1.00	4.006	6.060
6	2.304374	4.320	0.505	1.010	2.1816	1.00	4.807	7.272

3.3.2 Minimum Wire Cross Section

By using the second design procedure to find the cage with minimum wire cross section, the above design constraints will be satisfied with the design curves in figures 22 and 23. The results are given in table 5 for the 200-kV case.

The following conclusions follow from tables 4a and 5:

(1) Cages designed for minimum wire cross section have less area and windage for the same value of n than cages designed by the fixed wire size procedure.

(2) The two-wire-minimum cross-section cage satisfying the design criteria uses wires with 0.98-inch diameter and a separation of 14.8 inches.

(3) All the minimum cross-section cages have a cage radius at about 7.5 inches.

Table 5. Second design example—minimum wire cross-section wire diameter and cage radius for various values of n, 200 kV, h = 600 feet, a_{eqE} = 0.803 inch, $2h/a_{eqE} = 17,933$, E_0 = 1 kV/mm.

n	a_{eqE}/a	$b_{\min}/a_{eqE}$	a inch	d inch	b inch	a_{eqC}	E_n	Area inch2	Windage inch
1	1.000		0.803	1.606			1.00	2.026	1.927
2	1.640	9.200	0.490	0.979	7.388	2.689692	1.00	1.506	2.350
3	2.304	9.297	0.349	0.697	7.465	3.876951	1.00	1.145	2.509
4	2.980	9.477	0.269	0.539	7.610	4.668524	1.00	0.912	2.587
5	3.610	9.510	0.222	0.445	7.637	5.194605	1.00	0.777	2.669
6	4.290	9.557	0.187	0.374	7.674	5.570992	1.00	0.660	2.695
	Fig 22	Fig 23	0.803/col2	2*co14	0.803* col3		Eqn 5	n*Pi*a^2	1.2*n*d

3.4 THIRD DESIGN EXAMPLE (UMBRELLA TOP-LOADED MONOPOLE)

The objective of this example is to design appropriate cages for the ends of the top-load radials of a 1200-foot umbrella top-loaded monopole. The design requirements will be 250 kV and a maximum gradient of 1 kV/mm. The diameter of the wires is 1.01 inches.

The antenna corresponds to VLF Aguada and is illustrated in figure 26. It has 12 active top-load radials. These radials operate at high voltage. Each top-load radial is shielded by other top-load radials and the tower. Because all the shielding elements are at nearly the same voltage and most of the shielding elements are the same size, the effect of the shielding can be approximated by using the apparent height concept introduced in section 1.4.2. Once the apparent height has been determined, the design procedure is the same as for a single isolated wire.

An electrostatic computer code (Hansen & Simpson, to be published) has been used to obtain the charge distribution along one radial for a voltage of 100 kV (figure 27). The charge density is greatest near the ends of the wires where they are less shielded. Near the tower top, the charge density is small because the radials are well shielded by each other and by the tower. Without a corona ring at the end of the wire, the charge density increases, rapidly approaching twice the value it would be if the wire continued on. VLF Aguada has large corona rings at the wire ends that mitigate the end effect. Note that, for this case, a linear charge distribution is a good approximation.

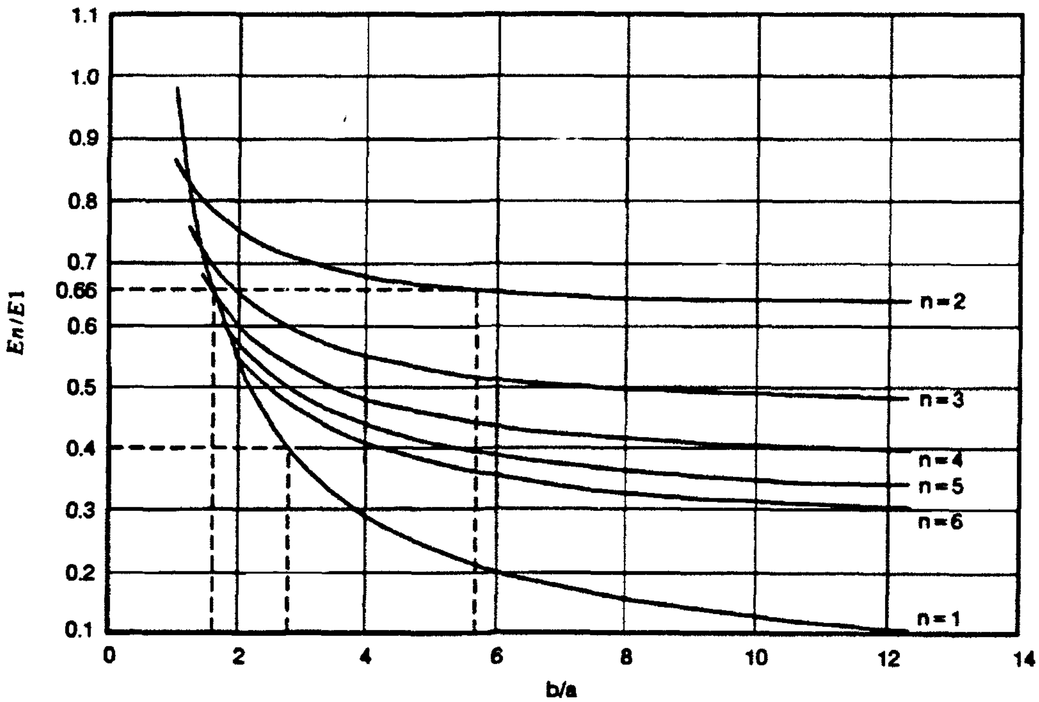

Figure 25. First design example, E_n/E_1 for various values of n ($2h/a$ = 28,515).

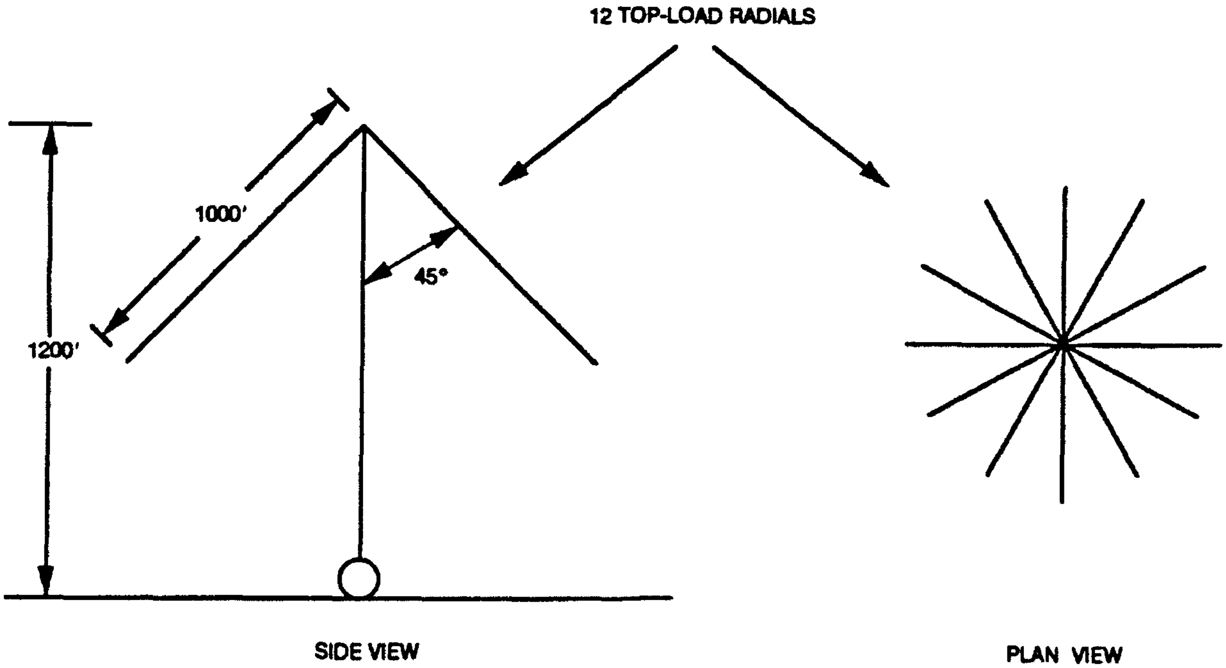

Figure 26. Umbrella top-loaded monopole.

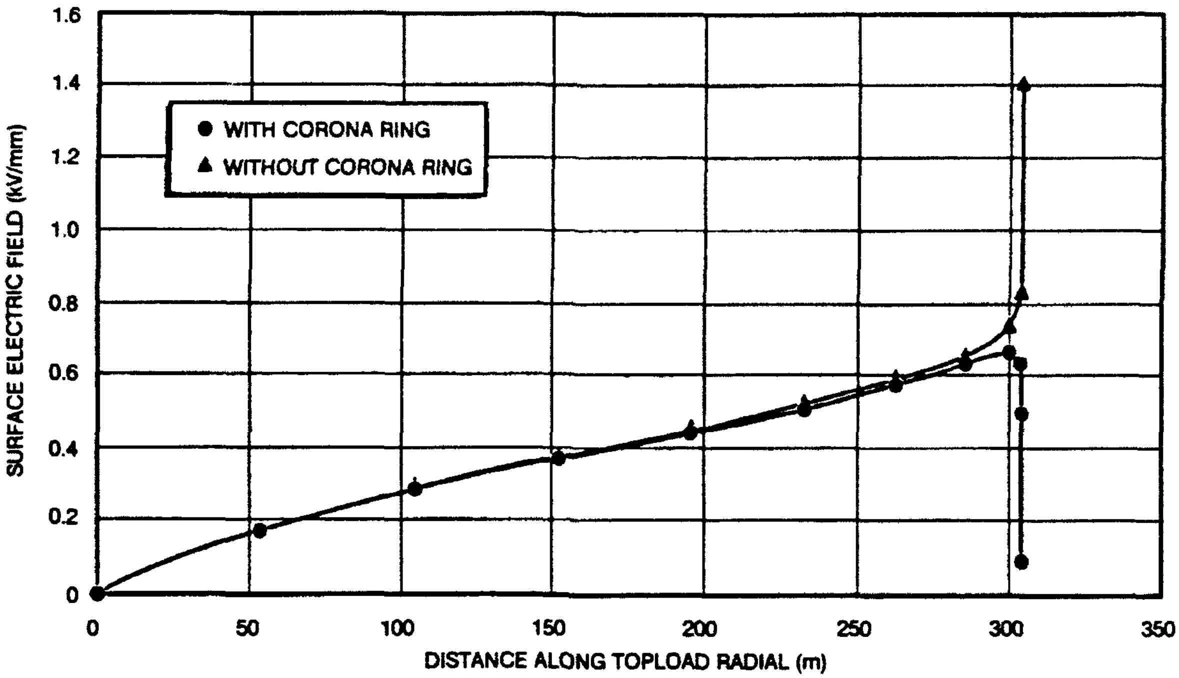

Figure 27. Aguada top-load surface gradient, 100 kV with and without corona ring.

The maximum surface electric field near the end of the radials (with corona rings) for 100 kV is 0.696 kV/mm.

To determine the apparent height for the high-gradient region near the end of the wires, find the height for a single isolated wire (same diameter and voltage) that gives the same gradient. For a wire diameter of 1.01 inches, 100 kV and a gradient of 0.696 kV/mm, the apparent height h_a is 1580 feet.

Note that the apparent height was also calculated by using the electrostatic code for wire diameters of 2 and 3 inches and found to be respectively 1813 feet and 1845 feet. The apparent height increases slightly with wire diameter for this geometry. If the value 1580 is used, the cage design will be slightly conservative in that the actual gradient will be slightly less than the calculated gradient.

3.4.1 Given Wire Size

Design cages that use the 1.01-inch-diameter wire meet the design criteria.

Using the value of h_a given above, a set of curves for E_m versus b/a were calculated using equation (5) (figure 28). The design voltage of 250 kV has been used. Note that in this figure, the gradient has not been normalized.

The curve for $n = 1$ is a special case for a single isolated wire and the ordinate for this case is wire diameter. The value of the diameter corresponding to a_{eqE} is determined from the value of the ordinate when $E_m = 1$ kV/mm, and has the value 1.854 inches.

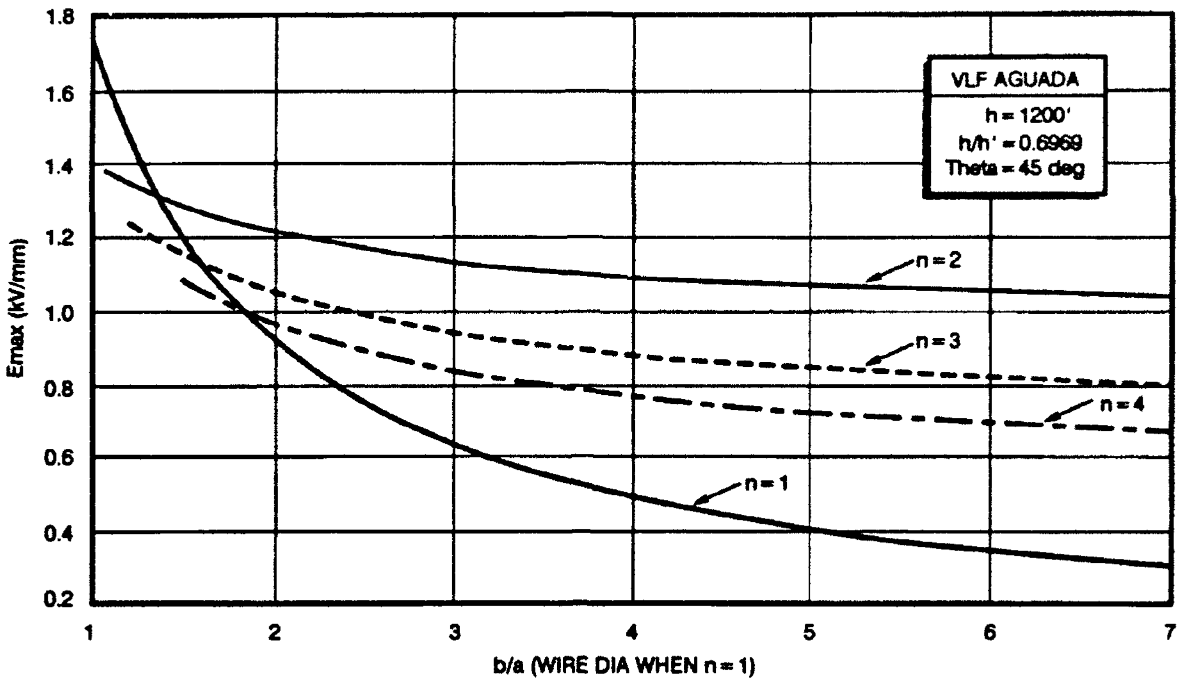

Figure 28. Umbrella top-loaded monopole 250 kV, 1.01-inch wires.

There is no solution for n = 2. Note that, for this case, the minimum gradient does come close to 1 kV/mm. Use of the design curve of figure 19, with a = 1.01 inch/2 and h_a = 1580 feet, gives b_{min}/a = 13. The optimum wire to wire spacing for the 1.01-inch wires is 13 inches. Substitution into equation (5) indicates that the minimum gradient for 250 kV is 1.09 kV/mm. Thus, reducing the voltage to 228 kV will satisfy the 1 kV/mm criteria.

There are solutions for n > 2. These solutions are summarized in table 6. From the table, the following conclusions can be drawn:

(1) The three-wire cage will satisfy the criteria if the spacing between wire centers is at least 2.1 inches.

(2) The four-wire cage is acceptable if the center to center spacing is at least 1.31 inches.

Table 6. Third design example—VLF Aguada with Vtop = 250 kV cages by using 1.0-inch-diameter wire, with E_{max} = 1 kV/mm, apparent height of wires = 1580 feet.

n	a inch	b/a	b inch	Wire spacing between centers S inch	E_{max} (kV/mm)	a_{eqC}
1	0.927				1.00	0.93
2	0.505					
3	0.505	2.4	1.212	2.10	1.00	1.31
4	0.505	1.84	0.9292	1.31	1.00	1.13

3.4.2 Minimum Wire Cross Section

The second design procedure is used to find the cage with minimum wire cross section, satisfying the design constraints. The design curves in figures 22 and 23 are used. The results are given in table 7 for 250 kV.

The following conclusions follow from table 7.

(1) The two-wire cage requires wires with diameter 1.11 inches, and wire-to-wire spacing of 18.7 inches.

(2) The required wire diameter for $n = 3$ is 0.792 inch and decreases with increasing n.

(3) For $n = 6$, the required wire diameter is less than 1/4 inch.

(4) The required cage radius (b_{min}) is about 9.5 inches for all cases.

Table 7. Third design example—minimum wire cross-section wire diameter and cage radius for various values of n, 250 kV, h_a = 1580 feet, a_{eqE} = 0.927 inch, $2h_a/a_{eqE}$ = 40,906, E_0 = 1 kV/mm.

n	a_{eqE}/a	b_{min}/a_{eqE}	a inch	d inch	b inch	a_{eqC}	E_n
1	1		0.927	1.854		0.927	1.00
2	1.67	10.05	0.555	1.110	9.316	3.216	1.00
3	2.34	10.16	0.396	0.792	9.416	4.723	1.00
4	3.04	10.28	0.305	0.610	9.526	5.698	1.00
5	3.71	10.32	0.250	0.500	9.567	6.367	1.00
6	4.39	10.38	0.211	0.422	9.622	6.863	1.00

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APPENDIX A

SINE PRODUCT IDENTITY

The sine product identity to be proven is given below,

$$\prod_{k=1}^{N-1} [2\sin(k\pi/N)] = N \quad . \tag{A-1}$$

Euler's identity for Sin(X) is given by

$$\mathrm{Sin}(X) = \{e^{ix} - e^{-ix}\}/(2i) \quad ,$$

where $i = (-1)^{1/2}$.

Substituting Euler's identity into the lefthand side of equation (A-1), and letting $X = k\pi/N$ gives

$$\prod_{k=1}^{N-1} [2\{e^{ix} - e^{-ix}\}/(2i)] \quad .$$

Operating on this result gives

$$= (1/i)^{N-1} \prod_{k=1}^{N-1} \{e^{ix} - e^{-ix}\} \quad ,$$

$$= (1/i)^{N-1} \prod_{k=1}^{N-1} e^{ix}\{e^{2ix} - 1\} \quad ,$$

$$= (1/i)^{N-1}\{\sum_{k=1}^{N-1} (-iX)\} \; x \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (1/i)^{N-1} \exp\{\sum_{k=1}^{N-1} (-ik\pi/N)\} \; x \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-i)^{N-1} \exp\{(-i\pi/N)(N(N-1)/2)\} \; x \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-i)^{N-1} \exp\{(-i\pi)((N-1)/2)\} \; x \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-i)^{N-1} (\exp\{-i\pi/2\})^{N-1} \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-i)^{N-1} (-i)^{N-1} \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-1)^{N-1} \prod_{k=1}^{N-1} \{e^{2ix} - 1\} \quad ,$$

$$= (-1)^{N-1} (-1)^{N-1} \prod_{k=1}^{N-1} \{1 - e^{2ix}\} \quad ,$$

and finally,

$$= \prod_{k=1}^{N-1} \{1 - e^{2ix}\} \quad ,$$

or

$$= \prod_{k=1}^{N-1} \{1 - e^{ik2\pi/N}\} \quad . \tag{A-2}$$

The set of exponentials,

$$e^{i2k\pi/N} \quad \text{for } k = 1 \text{ to } N-1 \ ,$$

are the Nth roots of 1 (except for 1 itself). These roots are sometimes called the cyclotomic numbers. The other Nth root of 1 is 1 itself. This can be seen by the following:

$$1 = e^{ik2\pi} \quad \text{for } k = \text{integer} \quad .$$

Taking the Nth root,

$$1^{1/N} = e^{(ik2\pi)/N} \quad .$$

These roots are represented on the complex plane by points on the unit circle at angles of $k2\pi/N$. There are a total of N unique roots (angles), because as k increases, the angle rotates around to a unique angle. When $k = N$, the angle is 2π, and the value of the root is 1. For $k > N$, the roots (angles) repeat.

These roots are solutions of the equation

$$Z^N - 1 = 0 \quad .$$

Using these roots to construct the equivalent polynomial gives

$$Z^N - 1 = (Z - e^{i2\pi/N})(Z - e^{i4\pi/N}) \cdots (Z -^{i(N-1)2\pi/N})(Z - 1) \ .$$

Rearranging terms gives

$$(Z-e^{i2\pi Pi/N})(Z-e^{i4\pi/N})\cdots(Z-^{i(N-1)2\pi/N}) = (Z^N-1)/(Z-1)$$

or

$$\prod_{k=1}^{N-1}(Z-e^{-ik2\pi/N}) = (Z^N-1)/(Z-1) \quad .$$

Note that the polynomial $Z^N - 1$ can be factored as follows:

$$Z^N - 1 = (Z-1)(Z^{N-1} + Z^{N-2} + \cdots + Z + 1) \quad .$$

Substituting into the above equation gives

$$\prod_{k=1}^{N-1}(Z-e^{-ik2\pi/N}) = (Z^{N-1} + Z^{N-2} + \cdots + Z+1) \quad .$$

Setting $Z = 1$ in the above equation gives

$$\prod_{k=1}^{N-1}(1-e^{-ik2\pi/N}) = N \quad .$$

Substitution of the above into equation (A-2) completes the proof.

APPENDIX B

INFINITE WIRE GRID CAPACITANCE

The geometry is an infinite set of wires parallel to ground and each other. They all have the same radius a and height above ground h. They are all separated by the same distance S. The geometry is illustrated in figure B-1. In the figure, the slant range between the image of one wire and other wires is denoted by D_k , where the subscript refers to the number of S spacings between the wires. D_k is given by the following equation.

$$D_k = \sqrt{(2h)^2 + (kS)^2} \ .$$

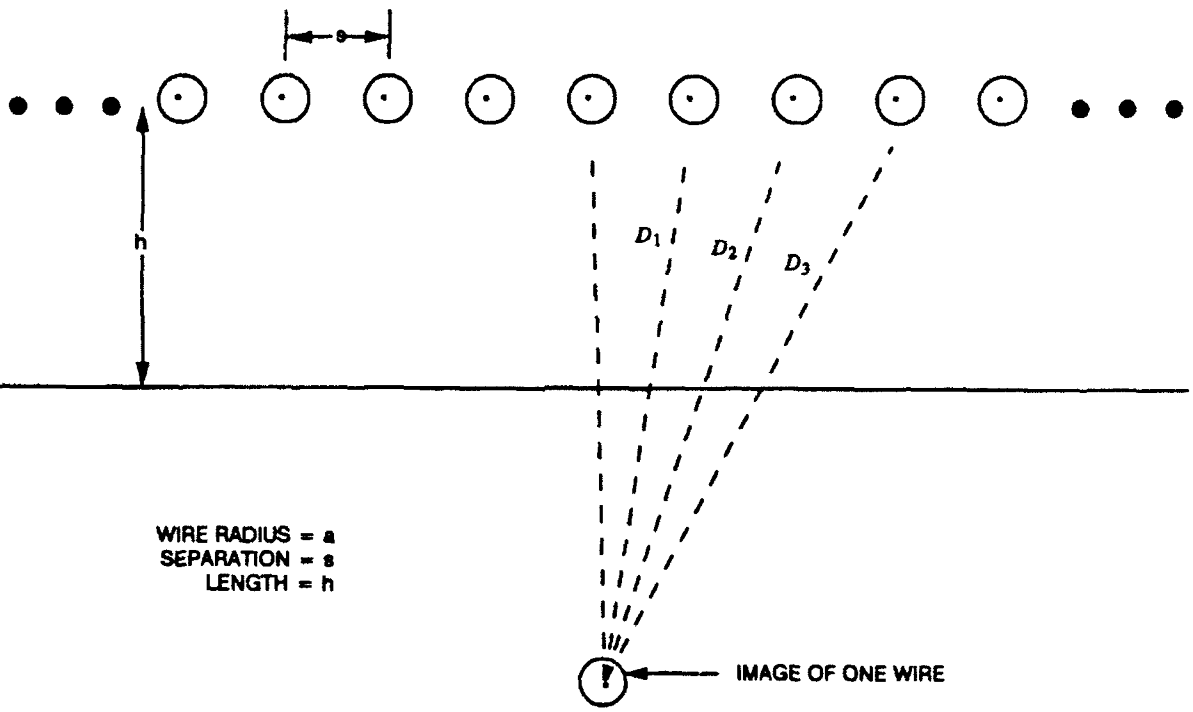


Figure B-1. Geometry for an infinite above-ground wire grid.

The wires all have the same voltage V and by symmetry they all have the same charge q. The potential at one wire is the sum of the potential from the charge on the wire and the potential due to the charge on all the other wires. This potential is given by the following equations:

$$\frac{2\pi\epsilon_0 V}{q} = \ln\left(\frac{2H}{a}\right) + \sum_{k=1}^{\infty} \ln\left(\frac{D_k}{kS}\right) \ ,$$

or

$$\frac{2\pi\epsilon_0 V}{q} = \ln\left(\frac{2h}{a}\right) + \sum_{k=1}^{\infty} \ln \sqrt{\frac{(2h)^2 + (kS)^2}{(kS)^2}} \quad ,$$

or, finally,

$$\frac{2\pi\epsilon_0 V}{q} = \ln\left(\frac{2h}{a}\right) + \frac{1}{2}\sum_{k=1}^{\infty} \ln\left[1 + \left(\frac{2h}{kS}\right)^2\right] \quad .$$

The summation term can be rewritten as a product

$$\frac{2\pi\epsilon_0 V}{q} = \ln\left(\frac{2h}{a}\right) + \frac{1}{2}\ln\left[\prod_{k=1}^{\infty}\left(1 + \left(\frac{2h}{kS}\right)^2\right)\right] \quad .$$

The infinite product in the above equation has a closed form. The derivation starts with the following series (Jolley, 1961):

$$\prod_{i=0}^{\infty}\left(1 - \left(\frac{\theta}{i}\right)^2\right) = \frac{\sin\theta}{\theta} \quad . \tag{B-1}$$

Solving for theta to make the products equal,

$$-\left(\frac{\theta}{\pi}\right)^2 = \left(\frac{2h}{S}\right)^2$$

$$\theta = \sqrt{-1}\ \frac{2\pi h}{S} \quad .$$

Substituting for theta in equation B-1, gives the closed form,

$$\prod_{i=1}^{\infty}\left(1 + \left(\frac{\theta}{i}\right)^2\right) = \frac{\sin\left(\frac{\sqrt{-1}\ 2\pi h}{S}\right)}{\frac{\sqrt{-1}\ 2\pi h}{S}} = \frac{\sinh\left(\frac{2\pi h}{S}\right)}{\left(\frac{2\pi h}{S}\right)} \quad .$$

Using the closed form gives the following equation for the capacitance per unit length of one of the wires.

$$C_1 = \frac{q}{V} = \frac{2\pi\epsilon_0}{\ln\left(\frac{2h}{a}\right) + \frac{1}{2}\ln\left(\frac{\sinh\left(\frac{2\pi h}{S}\right)}{\left(\frac{2\pi h}{S}\right)}\right)} \quad .$$

From the above equation, it is seen that each wire has the same capacitance it would have if it were elevated to a height of h_a.

$$h_a = h\left(\frac{\sinh\left(\frac{2\pi h}{S}\right)}{\left(\frac{2\pi h}{S}\right)}\right)^{1/2} .$$

APPENDIX C

IMAGE-THEORY FORMULATION

1. GENERAL

The method of images will be used to calculate the capacitance and electric fields in the vicinity of a set of parallel cylinders above a ground plane. The normal approach is to use a series of images to produce zero-potential on the surface of two cylinders. This method results in a nonconvergent infinite series. A convergent Image Theory will be developed by using dipoles. This approach accounts exactly for the field distortion due to the proximity of the cylinders, but does not account for the field distortion due to the ground plane. It is essentially exact when the cylinders are at a large distance above ground with respect to the radius of the wires.

2. THEORY

The development of the Image Theory starts with the simple problem of a line charge q parallel to a cylinder in free space.

The geometry for this problem is shown in figure C-1. It can be shown that the boundary condition (tangential $E = 0$), will be satisfied by using an image located at a point inside the cylinder boundary having the same magnitude but the opposite sign as the line charge. The image location is offset from the center of the cylinder toward the line charge, a distance given by

$$o = a^2/D,$$

where o is the offset, a is the radius of the cylinder, and D is the distance between the line charge and the center of the cylinder.

Since the cylinder must remain at neutral charge, another charge is placed at the center of the cylinder having the same sign as the line charge. The image charge plus the charge at the center form a dipole having no net charge.

The electric field at the point on the cylinder opposite the line charge is given by

$$E = \{q/(2\pi E_0)\}\ \{1/(D + a) + 1/a - 1/(a + o)\}\ .$$

Substituting for o gives

$$\begin{aligned} E &= \{q/(2\pi E_0)\}\ \{1/(D + a) + 1/a - 1/(a + o)\} \\ &= \{q/(2\pi E_0)\}\ \{1/(D + a) + o/(a^2 + ao)\} \\ &= \{q/(2\pi E_0)\}\ \{1/(D + a) + o/(a^2 + ao)\} \\ &= \{q/(2\pi E_0)\}\ \{2/(D + a)\}, \end{aligned}$$

which is exactly twice the value of electric field that would be at that position if the cylinder were removed.

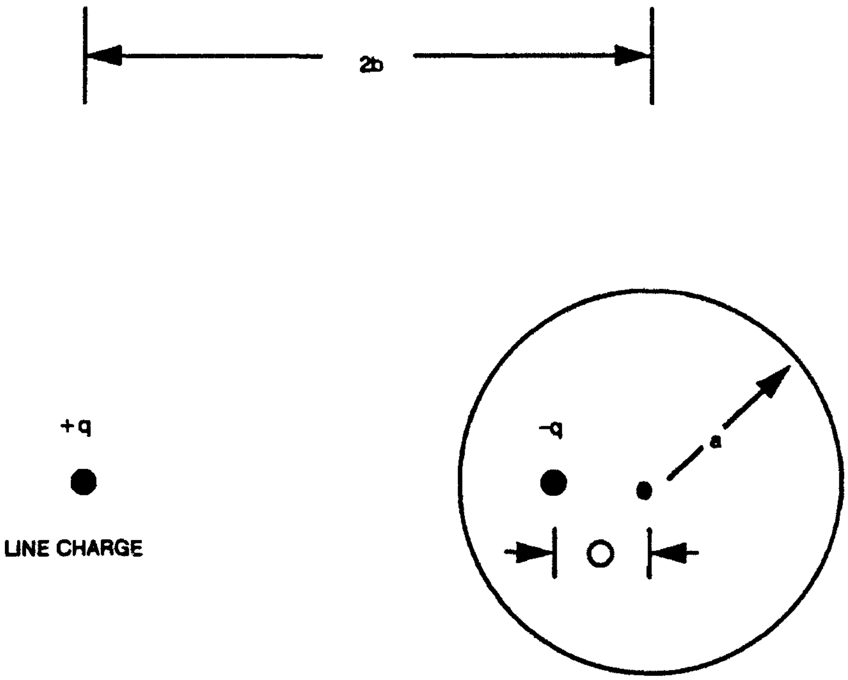


Figure C-1. Geometry for line charge and cylinder.

The series of images for the two cylinders, one charged and one neutral, is constructed by starting with a charge at the center of one cylinder, leaving the other cylinder neutral, and constructing an infinite series of dipoles. The first dipole is in the neutral cylinder and is the same as described above for the line charge cylinder combination with distance D being the same as 2b, the separation between cylinder centers. Each charge will be described by the sign charge and offset distance. For example, the charged cylinder starts out with a + charge located at the center (0 offset) designated by "+, 0." This gives rise to two charges on the neutral cylinder designated by "–, $a^2/(2b)$" and +, 0" (figure C-2).

Each of these charges gives rise to two charges on the charged cylinder. The charge +,0 gives –, $a^2/(2b)$, and +,0. While the charge –,$a^2/(2b)$ gives +,$a^2/(2b-a^2/2b)$ and –,0. Note that the two charges at the center of the cylinders (zero offset +,0 and –,0) cancel out. Also note that the offset for the next image of an image is a simple function of the offset of the image itself.

From the description and the figure, two important facts can be deduced: (1) all image charges at the center of the charged cylinder will cancel out and all image charges at the center of the neutral cylinder will cancel out except the first one; and (2) the offsets can be calculated by a recursion formula.

If the first offset is given by

$$o_1 = a^2/(2b),$$

then the other offsets are calculated by

$$o_i = a^2/(2b - o_{i-1}).$$

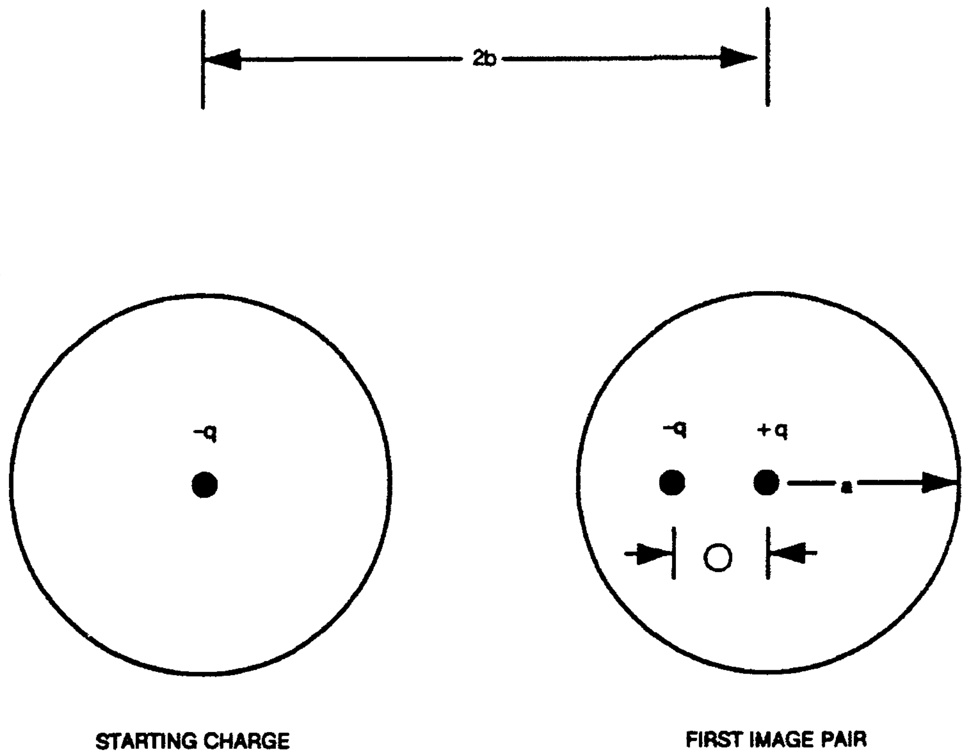


Figure C-2. Cylinder-to-cylinder images.

The distance between the offsets determines the dipole moment of that pair of charges. The series will converge if each succeeding image dipole moment is less than the source dipole moment. It can be shown that the offset converges to

$$\lim_{i \Rightarrow \infty} o_i = b \pm (b^2 - a^2)^{1/2}.$$

Since the offset converges to a fixed value, the distance between successive offsets approaches zero, and the series converges.

Table C1 illustrates the construction of the series, for both cases with one cylinder charged, the other cylinder neutral.

Table C1. Series of dipoles for the two-cylinder case. one cylinder charged, the other cylinder neutral.

Cylinder 1 (Charged)			Cylinder 2 (Neutral)		
First charge	+,0	=>	+,0	−,o_1	=>
−,o_1	+,o_2	=>	+,o_2	−,o_3	=>
−,o_3	+,o_4	=>	+,o_4	−,o_5	=>
. . .			. . .		
Cylinder 1 (Neutral)			Cylinder 2 (Charged)		
+,0	−,o_1	=>	First charge	+,0	=>
+,o_2	−,o_3	=>	−,o_1	+,o_2	=>
+,o_4	−,o_5	=>	−,o_3	+,o_4	=>
. . .			. . .		

From table C1, it is seen that the sign for charges at the even offset locations is +, while the sign for the charges at the odd offset locations is –.

When both cylinders are charged, the solution is by superposition, the sum of the charges on each cylinder for both cases given in table C1. To insure convergence, the charges must be taken in dipole pairs. Table C2 illustrates the sum of the two series.

Table C2. Sum of dipole pairs: both cylinders charged.

Cylinder 1 (Charged)		Cylinder 2 (Charged)	
	+2,0		+2,0
–2,o_1	+2,o_2	-2,o_1	+2,o_2
–2,o_3	+2,o_4	-2,o_3	+2,o_4
. . .		. . .	
–,o_i		–,o_i	

Several important points can be seen by examination of table C2.

(1) The series of charges are the same on both cylinders.

(2) The charge has a magnitude of 2 at every offset except for the last one used. The last charge location has a magnitude of 1, is always an odd offset and, therefore, is negative. This is a result of taking the charges in dipole pairs.

(3) The total charge on each cylinder is q, the originally assumed charge.

3. CAPACITANCE

The capacitance of the pair of cylinders is given by

$C = 2q/P,$

where q is the charge on one cylinder, and P is the potential of the cylinders (assumed to be the same for both cylinders).

The potential of a cylinder can be calculated by summing the potential at a point on the surface due to each charge. The point used to calculate the potential will be on the surface where the cylinders are closest together. The potential at this point, due to the charges on both cylinders at the ith offset location, is given by,

$$V_i = \left\{2q(-1)^i/(2\pi E_0)\right\}[\ln\{2h/(a-o_i\} + \ln\left\{((2h)^2 + (2b)^2)^{1/2}/(2b-a-o_i)\right\}] \; .$$

The total potential is calculated by summing the above terms for an even value of i, and then adding one more term $V_{i+1}/2$. To simplify interpretation of the results, the capacitance is often normalized to the capacitance of a single wire.

4. ELECTRIC FIELD

The maximum electric field will occur on the surface of the cylinders at the point where the cylinders are farthest apart. It is of interest to calculate also the minimum field that occurs on the surface, which will be at the location where the cylinders are closest together. The value of the field at these points, due to the charges on both cylinders at a particular offset, is given by

Maximum Surface Field

$$E_i = \{2q(-1)^i/(2\pi E_0)\}\ [(1/(a + o_i)) + (1/(2b + a - o_i)].$$

Minimum Surface Field

$$E_i = \{2q(-1)^i/(2\pi E_0)\}\ [(1/(a - o_i)) - (1/(2b - a - o_i)].$$

Again, the total field is given by summing the terms for an even value of i, and adding one more term, $E_{i+1}/2$, to account for the last term of the series.

To simplify interpretation of the results, the electric field is normalized to the value for a single wire with radius a. A simple BASIC computer program for calculating the normalized electric fields and the normalized capacitance is included as appendix F.

APPENDIX D

DERIVATION OF COMPLETE GEOMETRIC THEORY

The derivation for the complete geometric theory is the same as for the simple geometric theory except that the fields from the other wires are calculated on the surface of the wire instead of at the center of the wire. This leads to a more complicated geometric factor.

The assumptions used in this derivation are summarized below:

1. The charge on each wire is the same and located at the center of each wire;
2. The fields from the images of the wires in the ground plane are negligible; and
3. The fields in the vicinity of the wire where the field is being calculated are approximately uniform and, therefore, the effect of the wire can be accounted for by doubling the fields calculated as if the wire were not there.

The geometry for the general wire cage is redrawn in figure D-1. For convenience only, the first wire and jth wire are shown. The various angles involved are defined in the figure.

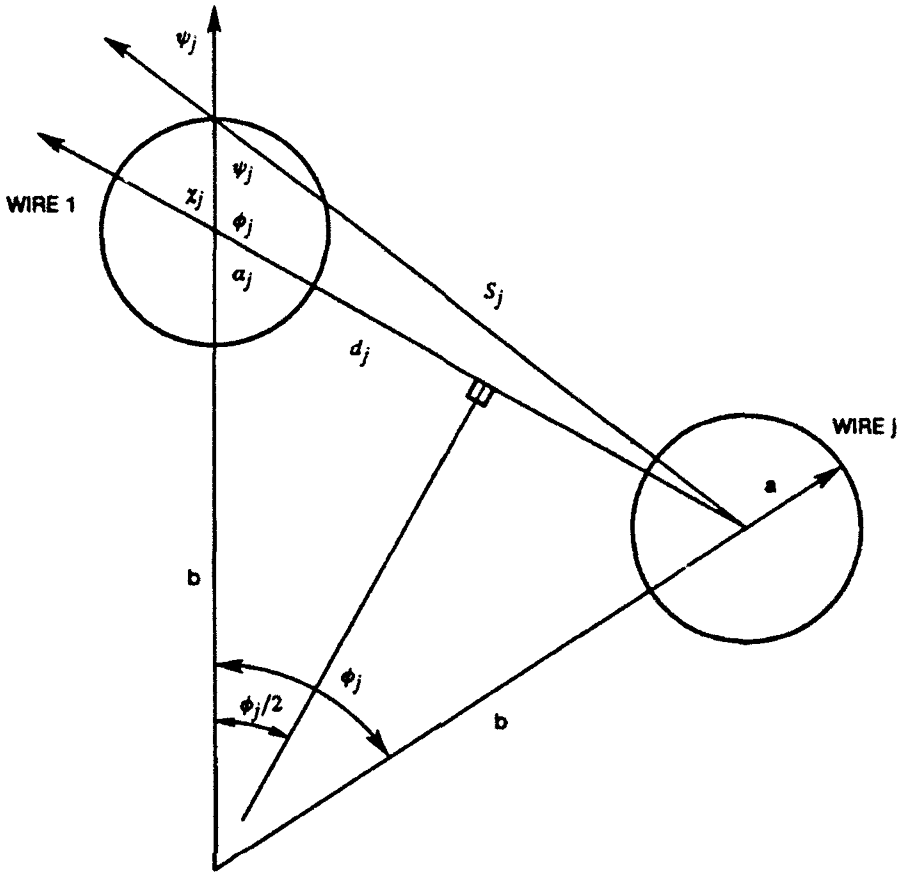
The field normal to wire 1 at the point opposite the cage center is given by

$$E_m = q\left[\frac{1}{a} + 2\sum_{j=2}^{n}\frac{\cos(\alpha_j)}{\rho_j}\right] ,$$

where ρ_j is the distance from the center of the jth wire to the field calculation point, and α_j is the angle between ρ_j and the radial from the cage center at the field calculation point.

The $1/a$ term represents the field on wire 1 from the charge on wire 1, and the summation term represents the field on wire 1 from each of the other wires. The factor of 2 in front of the summation accounts for the effect of wire 1 on the fields from all the other wires.

For this example, the fields are calculated on wire 1 but, by symmetry, all wires have the same field.



$$a_j = \pi/2 - \phi_j/2$$
$$\theta_j = \pi - \pi/2 + \phi_j/2$$
$$\chi_j = \pi - a_j = \pi/2 - \theta_j/2$$

Figure D-1. General cage geometry and angle definitions.

1. SIMPLE GEOMETRIC THEORY

For this theory, the fields are calculated at the center of the wire, i.e.,

$$\rho_j = d_j$$
$$\alpha_j = \sigma_j$$

(figure D-1),

$$\frac{\cos(a_j)}{(\rho_j)} = \frac{\cos(\chi_j)}{d_j} \quad .$$

Solving for each factor gives

$$\frac{d_j}{2} = b \sin\left(\frac{\theta_j}{2}\right)$$

or

$$d_j = 2b\sin\left(\frac{\theta_j}{2}\right)$$

and

$$\cos(\chi_j) = \cos\left(\frac{\pi}{2} - \frac{\theta_j}{2}\right) = \sin\left(\frac{\theta_j}{2}\right) .$$

Thus, each term of the series is the same

$$\frac{\cos(\chi_i)}{(d_j)} = \frac{1}{2b} ,$$

and the series sums to the following simple closed form

$$2\sum_{j-2}^{n} \frac{1}{2b} = \frac{n-1}{b} .$$

The final form of the equation is

$$E_m = \frac{q}{2\pi\epsilon_0}\left\{\frac{1}{a} + \frac{n-1}{b}\right\} .$$

Substitution of the total cage capacitance C_n gives the result

$$E_m = \frac{VC_n}{2\pi\epsilon_0 n}\left\{\frac{1}{a} + \frac{n-1}{d}\right\}$$

or

$$E_m = \frac{V}{n}\left\{\frac{1}{a} + \frac{n-1}{b}\right\}\left[\frac{1}{\ln(2h/a')}\right] .$$

This equation is called the simple geometric theory (SGT).

2. COMPLETE GEOMETRIC THEORY

For the complete geometric theory, the fields are calculated at the surface of wire 1, i.e.,

$$\rho_j = S_j$$
$$\alpha_j = \psi_j .$$

The summation becomes

$$2\sum_{j-1}^{n} \frac{\cos(\Psi_j)}{S_j} .$$

By using the definitions of figure D-1 and the law of Cosines, we can solve for S_j

$$S_j^{\,2} = d_j^{\,2} + a^2 - 2ad_j\cos(\phi_j) \quad .$$

Substitution of the following

$$\cos(\phi_j) = \cos\left(\frac{\pi}{2}+\frac{\theta_j}{2}\right) = -\sin\left(\frac{\theta_j}{2}\right)$$

$$d_j = 2b\sin\left(\frac{\theta_j}{2}\right)$$

gives

$$S_j^{\,2} = 4d^2\cos\left(\frac{\theta_j}{2}\right) + a^2 + 4ad\sin^2\left(\frac{\theta_j}{2}\right) \quad ,$$

which reduces to

$$S_j^{\,2} = a^2 + 4b(b+a)\sin^2\left(\frac{\theta_j}{2}\right) \quad .$$

Similarly, solving for $\cos(\psi_j)$

$$d_j^{\,2} = S_j^{\,2} + a^2 - 2aS_j\cos(\psi_j)$$

$$\cos(\psi_j) = \frac{S_j^{\,2} + a^2 - d_j^{\,2}}{2aS_j} \quad .$$

The summation terms are

$$\frac{\cos(\psi_j)}{S_j} = \frac{2a^2 + 2ad_j\cos(\phi_j)}{2aS_j} = \frac{a + 2b\sin^2\left(\frac{\theta_j}{2}\right)}{sj^2}$$

$$\frac{\cos(\psi_j)}{S_j} = \frac{a + 2b\sin^2\left(\frac{\theta_j}{2}\right)}{a^2 + (4b)(b+a)\sin^2\left(\frac{\theta_j}{2}\right)} \quad .$$

For $n = 2$ theta sub j over 2 = pi, and the equation for maximum surface field becomes

$$E_m = \frac{V}{2}\left\{\frac{1}{a} + \frac{1}{b+\frac{a}{2}}\right\}\left[\frac{1}{\ln(2h/a')}\right] \quad .$$

For n = 3 theta sub j over 2 = pi/3 and 2 pi/, and the equation for maximum surface field is

$$E_m = \frac{V}{3}\left\{\frac{1}{a} + 4\frac{a + 1.5b}{a^2 + 3ab + 3b^2}\right\}\left[\frac{1}{\ln(2h/a')}\right] .$$

For n = 4 theta sub j over 2 = pi/4, pi/2, and 3 pi/4, and the equation for maximum surface field is

$$E_m = \frac{V}{4}\left\{\frac{1}{a} + \frac{1}{b + \frac{a}{2}} + 4\frac{a + b}{a^2 + 2ab + 2b^2}\right\}\left[\frac{1}{\ln(2h/a')}\right] .$$

3. MODIFIED SIMPLE GEOMETRIC THEORY

The terms of the summation can be rewritten in the following form:

$$2\sum\frac{\cos(\chi_j)}{d_j} = \sum = \frac{1}{b + \frac{a}{2} + \frac{ab\left(\sin^2\left(\frac{\theta_j}{2}\right) - 1\right)}{a + 2b\sin^2\left(\frac{\theta_j}{2}\right)}} .$$

These terms are all of the form

$$\frac{1}{b + \frac{a}{2} + CF(\theta)} ,$$

where CF is a correction term. If we ignore this correction term, the series has a simple closed form, and gives the following equation for the maximum surface field

$$E_m = \frac{V}{n}\left\{\frac{1}{a} + \frac{n - 1}{b + \frac{a}{2}}\right\}\left[\frac{1}{\ln(2h/a')}\right] .$$

This equation is very similar to the simple geometric theory result and will be called the modified simple geometric theory (MSGT).

APPENDIX E

DERIVATION OF CHARGE OFFSET FORMULAS

The basis of this derivation is to assume the charge on each wire can be represented by a filament of charge located near, but offset from the center of the wire. The offset is in the direction of a line between the centers of the wires. The geometry is illustrated in figure E-1. The offset is determined by the requirement that the potentials calculated at two points on the surface of the wires are equal. The points are located where the extension of the line between the centers of the wires intersect the surface of the wire. These are the points where the wires are closest together and farthest apart. The offset is assumed to be in a direction such that the charge locations move towards the outside of the wires.

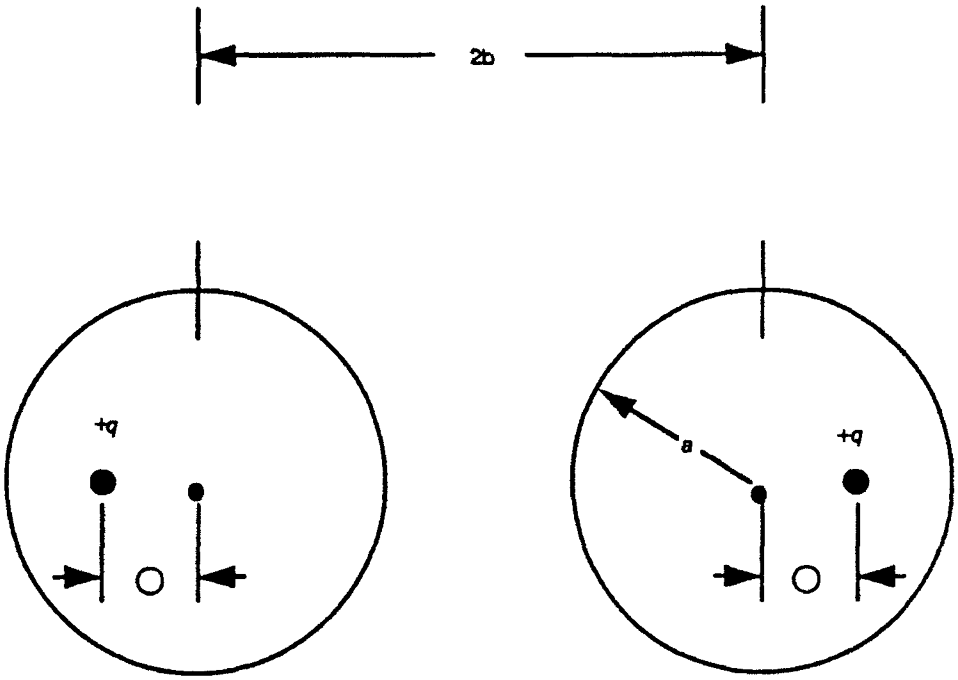


Figure E-1. Geometry for charge offset derivation.

The equations for the potential at these points are

$P_1 = \{q/(2\pi E_0)\}\ [\ln(2h/(a + o)) + \ln(2h/(2b + o - a))]$

$P_2 = \{q/(2\pi E_0)\}\ [\ln(2h/(a - o)) + \ln(2h/(2b + o + a))]$.

Setting these potentials equal leads to the following equation:

$\ln(2h/(a + o)) + \ln(2h/(2b + o - a)) = \ln(2h/(a - o)) + \ln(2h/(2b + o + a))$.

Eliminating the logarithms and reducing as follows results in a quadratic equation for o.

$(2h)^2/[(a + o)\ (2b + o - a)] = (2h)^2/[(a - o)\ (2b + o + a)]$

$(a + o)\ (2b + o - a) = (a - o)\ (2b + o + a)$

$2ab + ao - a^2 + 2bo + o^2 - ao = 2ab + ao + a^2 - 2bo - o^2 - ao$

$2o^2 + 4bo - 2a^2 = 0$

$$o^2 + 2bo - a^2 = 0. \qquad \text{(E-1)}$$

The quadratic equation for o has the solutions

$o = -b \pm (b^2 + a^2)^{1/2}.$

The capacitance of the pair of wires is given by

$C = 2q/P,$

where P is the potential, either P_1 or P_2 since by definition they are equal.

Substituting the value for the offset o into the equation for either P_1 or P_2 gives the following,

$P = \{q/(2\pi E_0)\}\ [\ln((2h)^2/(o^2 + 2bo - a^2 + 2ba)]$.

Substituting equation (E-1) gives

$P = \{q/(2\pi E_0)\}\ [\ln((2h)^2/(2ba)]$,

which gives the same potential as if the calculations were done with the charges located at the center of the wires. The capacitance for the two-wire case is the same as the SGT given by

$C = 4\pi E_0/[\ln((2h)^2/(2ba)]$,

$= 2\pi E_0/[\ln((2h)/(2ba)^{1/2}]$.

The maximum surface electric field, located at point 2 (figure E-1) is calculated by using Gauss' law and the value of the offset as follows,

$E = \{q/(2\pi E_0)\}\ \{1/(a - o) + 1/(2b + a + o)\}$.

Reducing the above equation gives

$E = \{q/(2\pi E_0)\}\ \{(2b + a + o + a - o)/((a - o)\ (2b + a + o))\}$,

$= \{q/(2\pi E_0)\}\ \{2(b + a)/(2ab + a^2 + ao - 2bo - ao - o^2)\}$,

$= \{q/(2\pi E_0)\}\ \{2(b + a)/(2ab + a^2 - 2bo - o^2)\}$.

Substitution of equation (E-1) further reduces this to

$E = \{q/(2\pi E_0)\}\ \{2(b + a)/(2ab)\}$,

$= \{q/(2\pi E_0)\}\ \{(b + a)/(ab)\}$,

$= \{q/(2\pi E_0)\}\ \{(1/a + 1/b)\}$.

$E = \{(1/(a - o) + 1/(2b + a + o)/\mathrm{n}\}\ [1/(\ln(2h/a'))]$.

Finally, substituting for q gives

$E = V\{(1/a + 1/b)/2\}\ [1/\ln(2h/a')]$,

which is exactly the same as the SGT for the two-wire case.

Note that dependence on the offset "o" explicitly cancels out in the formulas for both surface electric field and capacitance.

APPENDIX F

COMPUTER PROGRAM TO FIND b_{min} AND a_{eqE} IN BASIC

```
DECLARE SUB equiv (h2!, e0!, a0!)
        CLS
        OPEN "cage2.dat" FOR OUTPUT AS #1
        n = 2
        b = 1
        PRINT "Location of Minimum Gradient and Normalized Value"
        PRINT "n="; n
        PRINT "2h"; "      b"; "         Em"; "         aeqe"; "      aeqc"; "
        i = 100
        h2 = 0
        DO WHILE h2 < 100000
        h2 = h2 + i

        e0 = 100
100             aeqc = (n * b ^ (n - 1)) ^ (1 / n)
                emn = (1 + (n - 1) / (b + .5)) * LOG(h2) / n / LOG(h2 / aeqc)
                IF emn < e0 THEN
                        e0 = emn
                        b0 = b
                        aeqc0 = aeqc
                        b = b + 1
                        GOTO 100

                ELSE
                b = b0 - 1
                e0 = 100
                END IF

200             aeqc = (n * b ^ (n - 1)) ^ (1 / n)
                emn = (1 + (n - 1) / (b + .5)) * LOG(h2) / n / LOG(h2 / aeqc)
                IF emn < e0 THEN
                        e0 = emn
                        b0 = b
                        aeqc0 = aeqc
                        b = b + .1
                        GOTO 200

                ELSE
                b = b0 - .1
                e0 = 100
                END IF

300             aeqc = (n * b ^ (n - 1)) ^ (1 / n)
                emn = (1 + (n - 1) / (b + .5)) * LOG(h2) / n / LOG(h2 / aeqc)
                IF e0 >= emn THEN
                        e0 = emn
                        b0 = b
                        aeqc0 = aeqc
                        b = b + .01
                        GOTO 300
                ELSE
                END IF
                equiv h2, e0, a0

                PRINT h2; b0; e0; a0; aeqc; b0 / a0; b0 / aeqc
                PRINT #1, h2; b0; e0; a0; aeqc; b0 / a0; b0 / aeqc

                IF h2 = 1000 THEN i = 1000 ELSE
                IF h2 = 10000 THEN i = 10000 ELSE
                LOOP
```

```
                STOP
        END



SUB equiv (h2, e0, a0)
        a = 1
        em = e0
'       PRINT a, h2, e0

1000    em1 = LOG(h2) / a / LOG(h2 / a)
        IF em1 > e0 THEN
        a = a + .001
        GOTO 1000
        ELSE
        a0 = a

          END IF
        ' PRINT em1, a, e0

END SUB
```

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This report examines the parameters for systems of multiwire cages carrying high voltages above ground in order to minimize the maximum electric field on the surface of the wires. For the two-wire case, the following methods of calculation were compared: (1) a simple geometric theory (SGT); (2) a complete geometric theory (CGT); (3) an infinite series of images ("exact solution" for given constraints); and (4) a computer-based method of moments (MOM) solution. The MOM results were calculated for cages with 2 through 6 wires, and for 12 wires. This work resulted in two design procedures—one based on minimum wire spacing for a given wire diameter, and the other based on minimizing the total wire cross section of the cage. These procedures are presented along with three examples that include antenna and transmission line applications.

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