Q Calculations of L-C Circuits and Transmission Lines: A Unified Approach

Calculate the Q factor of any circuit based on its complex impedance data. These computations allow easy simulation and optimization of stub resonators. They apply to RLC circuits, transmission lines and antennas.

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Q factor calculations of reactive circuits are of interest since the Q factor relates directly to the circuit selectivity: The higher the Q, the better the selectivity and the lower the insertion loss of the filter. For oscillators, higher Q also means that lower phase noise is produced. In the case of antennas, a lower Q is generally preferred, giving a larger SWR bandwidth.

This paper attempts to clarify the various methods that can be used to calculate Q factor. I will show how to calculate the Q factor using general methods for simple RLC circuit configurations and will show that the same methods can be used to compute the Q factor of stub resonators and antennas operating outside their resonant frequencies. It allows computing the SWR bandwidth of antennas once the Q factor is known.

The calculations for transmission line stubs may be carried out using *Mathcad* files (described later in the article) with attenuation data provided by freeware programs available on the Web. Simulation results will be presented as well as measured Q values on a length of RG-58 cable.

Simple RX Circuits

When the internal circuit configuration or

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equivalent circuit is known, we can measure the complex impedance at a single frequency and calculate the Q of that impedance using the classic relation:

$$Q = \frac{|X|}{R_s} = \frac{R_p}{|X|} \qquad \text{(Eq 1)}$$

where X is the reactance, R_s the series resistance and R_p the parallel resistance.

The simple series model $R_s - jX$ assumes that the impedance consists of a resistance R_s in series with a *single* reactance X, coming from a perfect inductance or from a perfect capacitance. Note that the measurement of R_s and X only needs to be done at a single frequency. Examples of this are the Q of an ideal inductor or capacitor. In this model, only two elements are used to describe the impedance.

Since real capacitors all have some series inductance, we add it in series with the capacitor. At some frequency the reactances of the L and C will cancel and the impedance is reduced to the series R_s. Using Equation 1 to compute the Q factor yields a value of zero, since the total reactance is zero. In general, the above equation cannot be used to compute the Q factor when multiple reactances are involved. The Q factor as obtained from Equation 1 is called the apparent Q, since it is not generally related to the resonator selectivity.

In this last case the Q cannot be simply calculated by Equation 1. The equation does not tell us the selectivity of our circuit since

we now have a resonator. A resonator requires a minimum of two reactive elements that have opposite signs.

Q Factor of Series RLC Circuits

A simple RLC circuit — series or parallel — can be used as a resonator. We will first consider the series RLC configuration. Here we are interested in the Q factor at resonance and at other frequencies.

The impedance Z of the RLC circuit shows zero reactance at resonance. Then clearly Equation 1 cannot be used to compute the Q factor at resonance.

Equation 2 shows how to calculate the Q factor *at resonance and below*. Note that the derivative term

$$\frac{dX}{d\omega}$$

implies that the reactance X is effectively calculated (or measured) at two frequencies:[1, 2]

$$Q_a = \frac{-X + \omega\, \dfrac{dX}{d\omega}}{2R_s} \qquad \text{(Eq 2)}$$

where
$Z = R_s + jX$,
$R_s = \mathrm{Re}\left[Z \right]$
and
$X = \mathrm{Im}\left[Z \right]$

[1]Notes appear on page 51.

R_s and X and are the real and imaginary components of the RLC circuit impedance Z and $\omega = 2\pi f$.

Equation 2 may be simplified as:

$$Q_a = \frac{F_r}{R_s\,f}\sqrt{\frac{L}{C}} \qquad \text{(Eq 3)}$$

where Q_a is the Q factor below the resonant frequency F_r and f is the frequency at which Q_a is calculated.

The resonant frequency may be expressed as:

$$F_r = \frac{1}{2\pi\sqrt{LC}} \qquad \text{(Eq 4)}$$

After substituting Equation 4 into Equation 3, we get:

$$Q_a = \frac{1}{2\pi f C R_s} \qquad \text{(Eq 5)}$$

This equation is valid at resonance and below.

Equation 5 simply represents the ratio of the capacitive reactance to the series resistance, R_s.

An equation similar to Equation 2 may be used *above resonance*:

$$Q_b = \frac{X + \omega\dfrac{dX}{d\omega}}{2R_s} \qquad \text{(Eq 6)}$$

The sign of the first term of the numerator, X, is now positive. As before, Equation 6 may be simplified:

$$Q_b = \frac{f}{R_s\,F_r}\sqrt{\frac{L}{C}} \qquad \text{(Eq 7)}$$

After substituting Equation 4 into Equation 7, we get:

$$Q = \frac{2\pi f L}{R_s} \qquad \text{(Eq 8)}$$

This equation is valid at resonance and above.

Equation 8 simply represents the ratio of inductive reactance to the series resistance, R_s.

Equations 2 and 6 may be combined by taking the absolute value of X in the first numerator term. Equation 9 gives the Q factor of a series RLC circuit, below and above the resonant frequency.

$$Q = \frac{|X| + \omega\dfrac{dX}{d\omega}}{2R_s} \qquad \text{(Eq 9)}$$

This equation is for computing the Q factor above and below series resonance.

Figure 1 shows an example of the Q factor variation versus frequency as computed from Equation 9 or from Equations 5 and 8. Note that the Q factors calculated by Equations 5 and 8 are equal at the resonant frequency of 10 MHz, since the reactances are also equal. Above and below 10 MHz, the reactances increase, causing a corresponding increase in the Q factor. The "off resonance" Q factor gives the selectivity obtained when a lossless reactance is used to recover resonance.

Q Factor of Parallel RLC Circuits

The Q factor of RLC parallel circuits may also be calculated with the general formulas given by Equations 2 and 6 above. In this case, we need to use the admittances instead of the impedances, since the reactance goes to ±infinity at resonance, with a negative reactance slope. We then substitute $1/Z$ for Z in these equations.

As in series RLC circuits, the Q factor may also be calculated:

$$Q = \frac{-\,\mathrm{Im}\!\left[\dfrac{1}{Z}\right] + \omega\dfrac{d}{d\omega}\mathrm{Im}\!\left[\dfrac{1}{Z}\right]}{2\,\mathrm{Re}\!\left[\dfrac{1}{Z}\right]} \qquad \text{(Eq 10)}$$

where Re and Im are the real and imaginary operators.

This equation is for use below resonance, after substituting $1/Z$ for Z in Equation 2.

Equation 10 may be simplified:

$$Q_a = \frac{R_p}{2\pi f L} \qquad \text{(Eq 11)}$$

This equation is for calculating the Q below resonance. R_p is in parallel with L and C.

Above resonance:

$$Q = \frac{\mathrm{Im}\!\left[\dfrac{1}{Z}\right] + \omega\dfrac{d}{d\omega}\mathrm{Im}\!\left[\dfrac{1}{Z}\right]}{2\,\mathrm{Re}\!\left[\dfrac{1}{Z}\right]} \qquad \text{(Eq 12)}$$

This equation is for calculating the Q above resonance.

Equation 12 may be simplified:

$$Q_b = 2\pi\,f\,C\,R_p \qquad \text{(Eq 13)}$$

This equation is for calculating the Q above resonance.

Equations 10 and 12 may be combined by taking the absolute value of

$$\mathrm{Im}\!\left[\frac{1}{Z}\right]$$

in the first numerator term. Equation 14 gives the Q factor of a parallel RLC circuit, for use below and above the resonant frequency.

$$Q = \frac{\left|\mathrm{Im}\!\left[\dfrac{1}{Z}\right]\right| + \omega\dfrac{d}{d\omega}\mathrm{Im}\!\left[\dfrac{1}{Z}\right]}{2\,\mathrm{Re}\!\left[\dfrac{1}{Z}\right]} \qquad \text{(Eq 14)}$$

This equation computes the Q factor for parallel resonance.

Transmission Line Stub Resonators

The transmission line stub resonator always includes three distributed elements: inductance, capacitance and resistance. Taking a single measurement of the complex impedance at the stub terminals only allows us to represent the line as a simple two-element model, with an apparent Q value as given by Equation 1. It does not allow Q factor predictions under resonant conditions. While this is sufficient for some uses, very often one needs to know the stub Q factor when the stub is used as a resonator, with or without a compensating (loading) reactance. In particular, it is interesting to know the Q factor of a quarter wavelength resonator, versus one that is less than a quarter wavelength long and brought back to resonance by using capacitive loading at its open end. In some cases, the shorter resonator will have a higher Q. Thus it is useful to know how the resonator unloaded Q varies versus frequency, length,
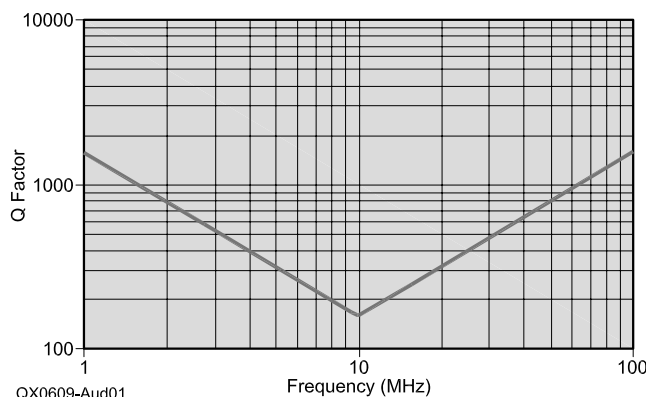

Figure 1 — *Q* factor of a series RLC circuit.

line attenuation and resonator loading such as capacitive or inductive loads in both open and short configurations.

Estimating the *Q* factor of a "Short" Unloaded Resonator

In general, Equation 1 above can't be used to compute the *Q* factor for a quarter-wave resonator, since the reactance value *X* is zero or infinite at resonance, and will set the *Q* to zero or infinity. We also expect a short line (say 1% of wavelength) to behave as lumped inductance in the case of a shorted line and a lumped capacitance in the case of an open line, and with a *Q* factor that can be approximated by Equation 1.

Computing the *Q* Factor of a Quarter Wavelength Unloaded Stub, Open or Shorted — The Easy Way

The simplest way to calculate the *Q* factor of a quarter wavelength unloaded stub is to use Equation 15.

$$Q = \frac{8.686 \ \pi}{A_o \ \lambda} \qquad \text{(Eq 15)}$$

where A_o is the attenuation in dB/100 ft and λ is the wavelength in hundreds of feet.

Equation 15 may also be written in a more practical form as:

$$Q = \frac{2.7743 \ F_o}{A_o \ VF} \qquad \text{(Eq 16)}$$

where F_o is the quarter wave resonant frequency in MHz, A_o is the attenuation in dB/100 ft at F_o and VF is the velocity factor. Equation 16 may also be derived from the attenuation coefficient α and the phase coefficient β:[3]

$$Q = \frac{\beta}{2 \alpha} \qquad \text{(Eq 17)}$$

I have found that Equations 15 to 17 apply to all unloaded resonators — open or shorted — whose length is an integer multiple of a quarter wavelength. Note that these equations do not distinguish between open and shorted quarter wave resonators. It was found that open and shorted resonators have equal *Q* factors for a given resonance mode, no matter their conductor and dielectric losses, as long as we use the same loss values for both lines.

From Equation 16, given constant values for A_o and *VF*, the *Q* factor is proportional to the frequency. In real life, the attenuation factor A_o will increase with frequency, causing the *Q* to increase less rapidly.

The line losses, A_o, are the total losses. Transmission lines have two loss mechanisms, however: conductor losses, which are caused by skin effect, and dielectric losses, which occur in the dielectric material.

In practice, in the HF and VHF ranges,

the dielectric losses are much lower than the conductor losses for coaxial lines. For those lines, we expect the *Q* factor of opened lines (subjected to dielectric losses) to be higher than their shorted equivalent (subjected to conductor losses) only when their length is much below a quarter wavelength.

Computing the *Q* Factor When the Stub Length is Below or Equal to a Quarter Wavelength

Computing the *Q* factor of a line requires knowledge of the attenuation due to conductor losses: A_oc in dB/100 feet and the attenuation due to dielectric losses —A_od in dB/100 ft. The frequency term, f, is in MHz.

$$A_oc = K_1 \ \sqrt{f} \qquad \text{(Eq 18)}$$

$$A_od = K_2 \ f \qquad \text{(Eq 19)}$$

where K_1 and K_2 are respectively the conductor and dielectric losses in dB/100 ft at 1 MHz.

The K_1 and K_2 loss coefficients may be obtained from the TLDetails.exe program.[4] It gives the coefficients for most coaxial cables. An accompanying Microsoft *Excel* file also allows K_1 and K_2 calculations for user-entered attenuation data.

For printed circuit lines such as microstrips, it is convenient to express the losses as follows:

$$A_oc = A_ocr \sqrt{\frac{f}{f_r}} \qquad \text{(Eq 20)}$$

This equation calculates attenuation due to conductor losses.

$$A_od = A_odr \ \frac{f}{f_r} \qquad \text{(Eq 21)}$$

This equation calculates attenuation due to dielectric losses.

$$A_o = A_oc + A_od \qquad \text{(Eq 22)}$$

This equation calculates the total losses. A_o is the total attenuation in dB/100 ft, A_ocr is the conductor loss attenuation in dB/100 ft at frequency f_r, f is the frequency, A_odr is the dielectric loss attenuation in dB/100 ft at the same frequency, f_r.

Note that the conductor losses vary in proportion to the square root of the frequency, and the dielectric losses are proportional to frequency. The program TXLine.exe may be used to compute the PCB line attenuations for various configurations.[5]

Note that A_od may also be derived from the loss tangent, *L tan*, of the dielectric. *VF* is the velocity factor and f is the frequency in MHz.

$$A_od = 2.78 \ L \ tan \ \frac{f}{VF} \qquad \text{(Eq 23)}$$

We need to compute the attenuation co-

efficient, α, and the phase coefficient, β:

$$\alpha = \frac{\ln(10)}{2000} \ A_o \qquad \text{(Eq 24)}$$

(Attenuation in neper/foot)

$$\beta = \frac{2 \ \pi \ f}{VF \ c_e} \qquad \text{(Eq 25)}$$

(Phase coefficient in radians/foot)

VF is the line velocity factor, c_e is the velocity of light in million feet/sec, A_o is the total attenuation in dB/100 ft and f is the frequency in MHz. The complex propagation coefficient, γ, may now be calculated:

$$\gamma = \alpha + j\beta \qquad \text{(Eq 26)}$$

To properly model the lossy transmission line, we must compute the complex line impedance, Z_o, using Equation 27. Here the term *R* represents the series conductor losses due to the skin effect and dc conductor resistance. *G* is a conductance term representing the parallel dielectric losses. Both *R* and *G* are a function of the frequency, *f*, thus making Z_o a complex value, which is also a function of the frequency.[6]

$$Z_o = \sqrt{\frac{R + j2 \ \pi \ f \ L}{G + j2 \ \pi \ f \ 10^{-6} \ C}} \qquad \text{(Eq 27)}$$

In Equation 27, *R* and *G* are in ohms per foot and siemens per foot, respectively. The *L* and *C* are the distributed inductance in μH/foot and capacitance in pF/foot, respectively. The frequency, *f*, is in MHz.

The *R, L, C* and *G* components will be calculated as follows (See Note 6.):

$$L = \frac{\text{Im}\left[\gamma \ Z_o \right]}{2 \ \pi \ f} \qquad \text{(Eq 28)}$$

L is in μH per foot.

$$C = \frac{\text{Im}\left(\dfrac{\gamma}{Z_o} \right)}{2 \ \pi \ f \ 10^{-6}} \qquad \text{(Eq 29)}$$

where *C* is in pF per foot.

$$R = 2 \ \alpha_c \ \text{Re}\left[Zo \right] \qquad \text{(Eq 30)}$$

where *R* is in ohms per foot.

$$G = \frac{2 \ \alpha_d \ \text{Re}\left[Zo \right]}{\left| Zo \right|^2} \qquad \text{(Eq 31)}$$

where *G* is in siemens per foot

Computing *R* and *G* requires the knowledge of Z_o and to compute Z_o we need the *R* and *G* values. To get around this problem we use an iterative process where we first use a real value of $Z_o + j \ 0$ ohms for Z_o (the cable nominal impedance). We then calculate *R, L, G* and *C*. These values are then used to recalculate a new complex value for Z_o. This
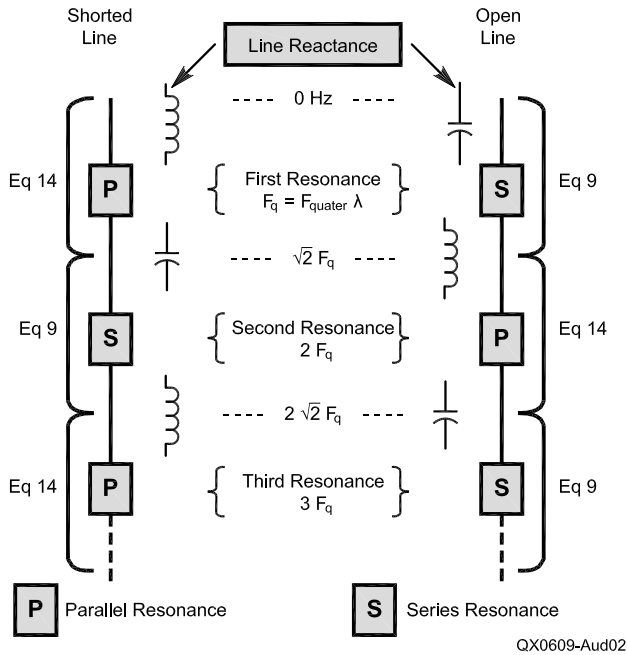
Figure 2 — Resonant modes for shorted and open lines along with the relevant equations for *Q* calculations.
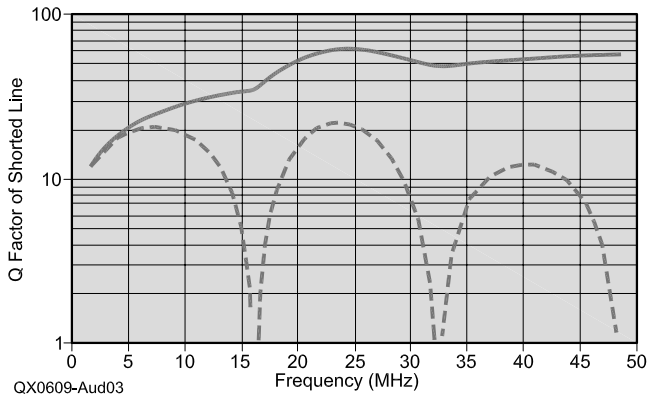

Figure 3 — Q factor of a shorted line. Q factor versus frequency for a 10 foot length of RG-58C, giving first resonance at ~16.229 MHz (*f*_q). Solid line is resonator *Q*, dotted line is the apparent *Q*.
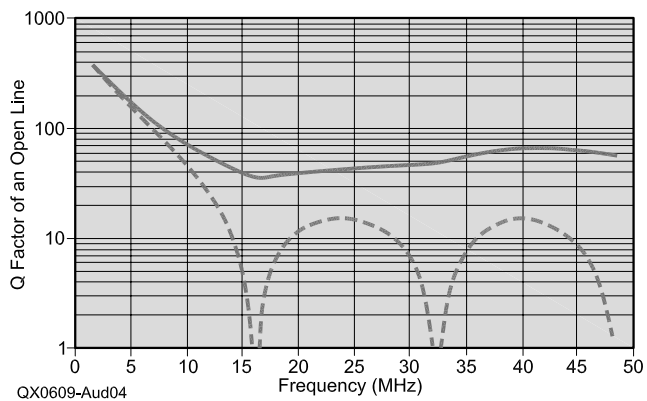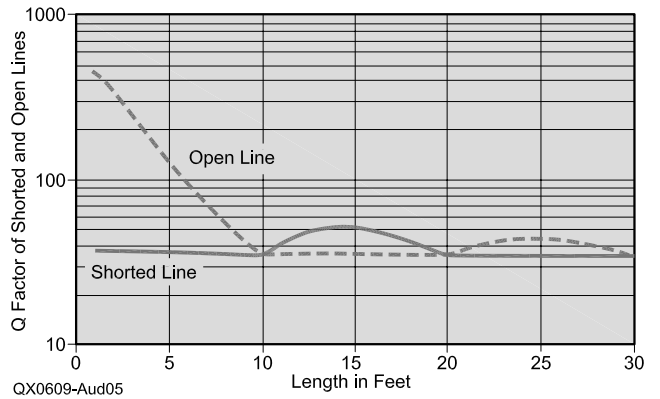

Figure 4 — *Q* factor of an open line. *Q* factor versus frequency for a 10 foot length of RG-58C, giving first resonance at ~16.229 MHz (*f*_q). Solid line is resonator *Q*, dotted line is the apparent *Q*.

Figure 5 — *Q* factor versus line length for RG-58C, at 16.229 MHz.
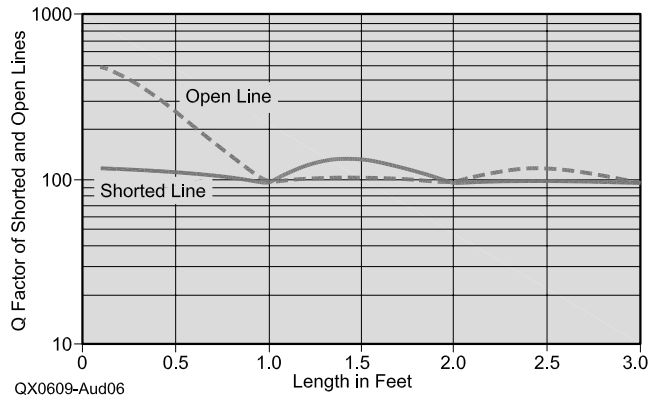
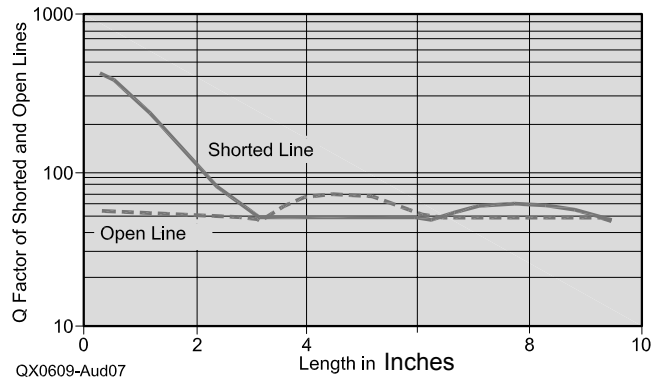Figure 6 — *Q* factor versus line length for RG-58C, at 162.29 MHz.

Figure 7 — *Q* factor versus line length at 500 MHz for a 50 Ω microstrip, 114 mil wide, above a 62 mil thick FR4 substrate.

process is repeated twice until we get a final value for Z_o.

We are now ready to calculate the stub impedances, using the complex value of Z_o, for both open and short lines. Equation 32 or 33 will be used to calculate the stub impedances.

$$Z_s\,open = \frac{Z_o}{\tanh(\gamma\ len)} \qquad \text{(Eq 32)}$$

where *len* is the line length in feet and Z_o the stub line complex impedance.

The shorted stub impedance may be calculated as:

$$Z_s \, short = Z_o \, \tanh(\gamma \, len) \qquad \text{(Eq 33)}$$

The stub Q factor may now be calculated as a function of frequency or length using Equation 9 for an open stub, since it behaves like a series resonant circuit. For a shorted stub, we use Equation 14 to calculate the Q factor, just like in the case of the RLC parallel circuit. These calculations are valid below and above the quarter wave resonant frequency.

Resonant Modes

Figure 2 shows the resonant modes for shorted and open lines. Equation 14 is used when the line exhibits parallel resonance and Equation 9 when it exhibits series resonance, just like for discrete RLC resonators. Note that the shorted line presents an inductive reactance below the first quarter wave resonance while the open line is capacitive below resonance.

The *Mathcad* spreadsheets TRL_Q_Calc1.mcd for use with the TLDetails.exe program on coaxial lines and TRL_Q_Calc-PCB1.mcd for use with TXLine.exe program on PCB lines show all above calculations in detail.[4, 5, 7]

Examples of Calculated *Q* factors versus Frequency for Shorted and Opened Lines of Identical Lengths

Figures 3 and 4 show the resonator Q factor for shorted and open lines (a 10-foot length of RG-58C). The solid curves were computed from Equations 14 and 9 as per Figure 2, while the dotted curves show the ratio of reactance to resistance as computed by Equation 1. This is the apparent Q. In general, the resonator Q cannot be computed just by taking the ratio of reactance to resistance as in Equation 1. This
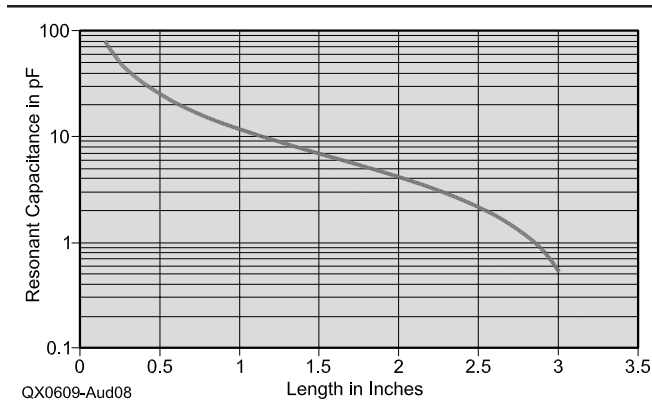

Figure 8 — Capacitance in pF required to keep the resonant frequency at 500 MHz.
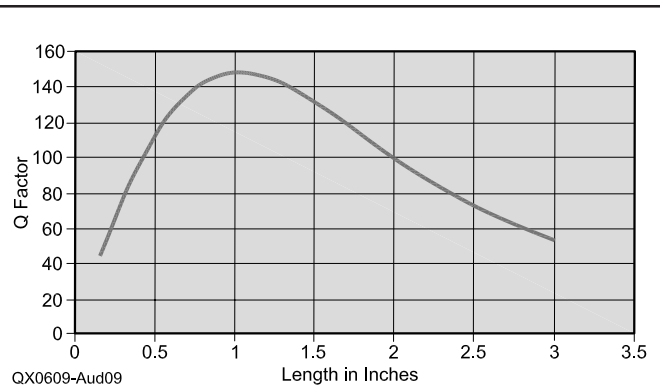

Figure 9 — Shorted stub *Q* factor versus length with the microstrip resonated with a capacitor having an ESR of 0.08 Ω.

Table 1

Mode	Freq (MHz)	Shorted Line EQ. Used	Calc Q	Q error	Open Line EQ. Used	Calc Q	Q error	Conductor Losses dB/100 ft	Dielectric Losses dB/100 ft
1X	10	14	28	<0.1%	9	79	<0.1%	1.0	0.1
1X	16	14	18	0.2%	9	29	<0.1%	2.0	0.02
1X	16	14	16	0.2%	9	14	<0.1%	1.0	1.00
1X	16	14	166	<0.1%	9	142	<0.1%	0.1	0.10
1X	18.5	14	15.5	0.1%	9	22.8	0.1%	2.0	0.02
1X	19	14	10	0.5%	9	11	0.6%	4.0	0.02
1X parallel	19.8545	14	20.3	0.2%				2.0	0.02
1X series	19.8773				9	20.4	<0.1%	2.0	0.02
2X	21	14	23.4	0.1%	9	21	0.2%	2.0	0.02
2X	28	14 or 9	36.9	0.3%	9 or 14	24.3	0.6%	2.0	0.02
2X	28	14 or 9	14.1	2.4%	9 or 14	17.5	1.6%	1.0	1.0
2X	28	14 or 9	143.7	<0.1%	9 or 14	177	<0.1%	0.1	0.1
2X parallel	39.729	9	28.6	0.3%				2.0	0.02
2X series	39.71				14	28.6	0.2%	2.0	0.02
3X	41	9	29	0.3%	14	30.8	0.2%	2.0	0.02
3X	50	14	15.7	4.0%	9	13.4	5.3%	1.0	1.0
3X	50	14	139.6	<0.1%	9	162.4	<0.1%	0.1	0.1
3X parallel	59.5863	14	34.6	0.4%				2.0	0.02
3X series	59.609				9	34.6	0.2%	2.0	0.02

approximation is valid for frequencies below 25% of the quarter wave resonant frequency (~16.229 MHz), however. For both shorted and open stubs, Equation 16 may be used to calculate the Q factor at all integer multiples of a quarter wavelength.

Using a constant real value for the line impedance makes the Q factor versus frequency equal for both open and shorted lines. This method makes calculations much faster and simpler, but as shown in Figures 3 and 4, it will give very large errors in the Q and in the complex stub impedance. I also discovered that the common transmission line models used in my professional RF-microwave circuit simulator use this shortcut too.

Note the Q factor behavior below the quarter wave frequency F_q. At frequencies below F_q, the Q factor goes down for the shorted line while it goes up for the open line. This tells us that the line losses are mostly in the conductors (~1.8 dB/100 ft while the dielectric losses are ~ 0.14 dB/100 ft. at the quarter wave resonant frequency).

It is also interesting to compute the Q factor versus line length at a fixed frequency. Figure 5 shows the Q factor at 16.229 MHz for a 10-foot length of RG-58C cable. Again the open line has a much higher Q factor below the quarter wave resonant length of 10 feet.

Figure 6 shows the Q factor versus line length for a 1 foot length of RG-58C line. Decreasing the line length by a factor of 10 has increased its resonant frequency by the same factor and the Q at self resonance goes from 36 to 100, a factor of ~3. This is possible since the losses are mainly conductor losses: 5.77 dB/100 ft and the dielectric losses are 1.36 dB/100 ft at 162.29 MHz. Note also that the loss tangent of the dielectric is 0.002.

In contrast, PCB losses will be much higher with say, FR4 which has a typical loss tangent of 0.02

The Q factor for a 50-Ω microstrip trace has been computed in Figure 7. The PCB loss tangent is 0.02 and the trace length is 3.469 inches to obtain quarter wave resonance at 500 MHz. At that frequency, the conductor losses are 5.7 dB/100 feet and the dielectric losses are 47.5 dB/100 feet.

Note that the open line now has its Q factor almost constant below quarter wave resonance. The Q factor of the shorted line is now much higher, in fact the shorter the line the higher the Q. The shorted stub becomes attractive as a resonator by using shorter lines with the higher Q. Then a low loss capacitor will be required to bring the line back to resonance at 500 MHz.

Figure 8 shows the required stub loading capacitance versus line length to resonate at 500 MHz. When the line length approaches a quarter wavelength, the required loading capacitance decreases toward zero.

Real capacitors have a series resistance that limits their Q factor, however. In the next example, a series resistance of 0.08 Ω is assumed. The effective Q factor of the stub and the capacitor may be computed by recalling that the Q factors add like parallel resistors. There is now an optimum length that will provide the highest Q, which is around 1 inch or about 25% of the original self resonant length. See Figure 9.

The highest Q will generally be obtained by paralleling multiple capacitors to decrease the effective series resistance. Note that wider PCB traces will provide higher Q factors, even if the line Z_o is lower.

Substituting RG-174 coax gives a maximum Q of 100 at 2.5 inches, while a length of RG-213 shows a maximum Q of 394 at a quarter wavelength (3.9 inches). This means that a low to very-low-loss line will have its highest Q at a quarter wavelength and above. I have measured the unloaded Q of a 6-inch-diameter quarter-wave resonator at 145 MHz in quarter-wave mode and 430 MHz in ¾-wave mode. At 145 MHz, the unloaded Q was 5324 and 9065 at 430 MHz: an increase by a factor of 1.7. This is also predicted by the models presented here.

Note that adding a lossless reactance in parallel (or series) with the stub — to modify its resonant frequency — does not change its Q versus frequency as calculated from Equations 9 or 14. Only its resonant frequency is changed.

So far from these simulations, Equations 9 and 14 make sense for calculating the resonator unloaded Q for resonator lengths ranging from 1% of the wavelength to many wavelengths. At low frequencies where the resonator is less than $^1/_{16}$ wavelength, we get the same results by simply taking the reactance-to-resistance ratio to compute the Q factor instead of using the more complex Equations 9 and 14.

Validating the Computed Resonator Q from the Calculated Stub Impedance

A second method of calculating the resonator Q factor has been used to determine the limits of validity of Equations 9 and 14 as applied to transmission lines. The resonator Q may be determined by finding the two frequencies f_1 and f_2 around the stub resonance that yield a stub impedance with a phase angle of ±45°. The exact stub resonant frequency is f_r, where its input reactance is zero in the case of series resonant lines. For lines that are parallel resonant, we use admittances and note the frequency where the suceptance goes to zero. For Q factors above 10, f_r can be taken as the average of f_1 and f_2.

The stub Q factor can then be calculated from:

$$Q = \frac{f_r}{f_2 - f_1} \qquad \text{(Eq 34)}$$

Note that the stub impedance measurements may be done at either end of the line, without affecting the Q value.

This method may also be used to compute the Q factor at frequencies other than at resonance by adding a lossless frequency-dependant series reactance (L or C) in the case of a series resonant line. Similarly, parallel resonant lines require the addition of a susceptance in shunt with the line. This method was used to verify the accuracy of Equations 9 and 14 with the help of *Mathcad* to do the calculations.

Table 1 summarizes the results for a line length of 8.17 feet. The frequency for the losses is 5 MHz. The Q error was derived by using Equation 34 to compute the resonator Q from the impedance/admittance data and comparing the values obtained with Equations 9 or 14.

At 28 MHz, we are about halfway between resonances, and either Equations 9 or 14 may be used, for both shorted and opened lines. This value is the geometric average between the first and second resonances.

As it can be seen from Table 1, the error is largest for Q values below 20 or so. The worst errors occur at the highest resonance modes having the lowest Qs. Below quarter wave resonance, the Q error is below 1% for Qs above 10.

Computing Resonator Q from Bandwidth Measurements at Multiples of Quarter Wavelength Resonance

Another way to verify the resonator Q factor is to build a band-pass filter and measure its selectivity by measuring its –3 dB points on an S21 display. To precisely determine the resonator Q_u (unloaded Q), the coupling of the stub under test to the source and detector will have to be very small. Equation 34 is used to compute the resonator Q factor. Figure 10 shows the circuit that I have used for my simulations on a line, which behaves as a parallel LC resonator.

In Figure 10, the source and detectors have low impedance (50 Ω). The stub is coupled to the source-detector via 0.1 pF capacitors. This method is useful to determine the resonant frequency of a shorted quarter wave line, since it presents the highest impedance at resonance. Again, Equation 34 is used to compute the stub unloaded Q factor, where f_1 and f_2 are the –3 dB frequencies, as measured from the resonant frequency f_r. Note that the source and load resistive impedances will decrease the Q factor somewhat. Equation 35 may be used to calculate the unloaded

Q, based on the measured attenuation at the resonant frequency and percent bandwidth.

Computing Resonator *Q* from Bandwidth Measurements below Quarter Wavelength Resonance

Figure 11 shows the setup that may be used with series resonant RLC elements when the resonant line is inductive or capacitive. The total reactance is cancelled, leaving the series R_s at resonance. This series R_s is easily determined by doing an attenuation test (S21) in a 50-to-50 Ω circuit at the series resonant frequency. The equivalent L and C lumped elements must also be determined by doing two more attenuation tests, before one can calculate the *Q* factor. I have developed an *Excel* file that does these calculations: Calc_SeriesRLC.xls.[7] Note that the C element has to be replaced by an inductor when the stub impedance is capacitive.

Tests were done on a 100-inch length of RG-58 cable, by measuring the complex reflection coefficient over 201 frequency points, using a lab-quality vector network analyzer (HP 8753D). See Figure 12. The measured coefficients were then converted to impedance data using *Mathcad*. The *Q* factor was calculated as per Equations 9 and 14 (solid curves) and as per Equation 1 (dotted curves). These dotted curves show the cable resonant frequencies; for example, where the reactance is zero: 19.4 MHz and 38.8 MHz.
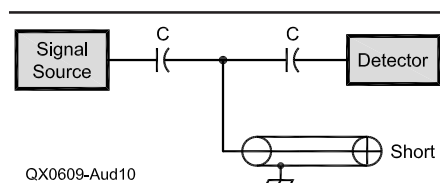

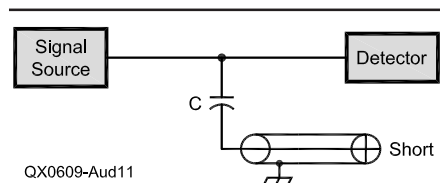**Figure 10 — Measuring and computing resonator *Q* from bandwidth measurements.**

Figure 11 — Measuring and computing resonator *Q* below quarter wavelength resonance.

Shorted and Open Line Measurements

In Figure 12, the shorted line exhibits parallel resonance at 19.4 MHz, and Equation 14 is used to compute the resonator *Q* factor up to 27 MHz. Equation 9 is used from 27 MHz up to 50 MHz, since we have a series resonance at 38.8 MHz.

For the open line of Figure 13, the equations are used in the reverse order. Note that the open line has a higher *Q* factor. The above *Q* curves are representative of a line that has much less dielectric losses than conductor losses as in Figures 3 and 4. Note the *Q* curve sloping down below 3 MHz. This is possibly caused by the inaccuracies in the measurements, especially for resistive impedance values below 5 Ω.

Measured *Q* Values versus Computed Values

Table 2 shows the percentage error between measured *Q* factors versus the computed values at three frequencies. Notice that there is good agreement between the two. The measured *Q* was computed directly from attenuation measurements, as per Figures 10 and 11. A series inductor was used to resonate the line when its reactance is capacitive and vice-versa.
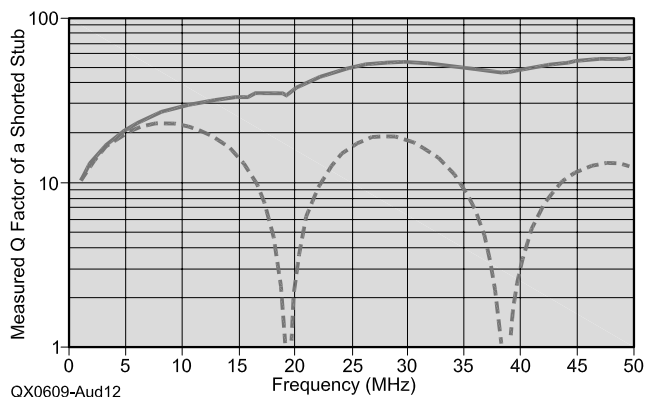

Figure 12 — Measured *Q* factor of a shorted line. Computed *Q* factor versus frequency (solid lines, from impedance measurements) for a 100 inch length of RG-58, giving first resonance at ~19.4 MHz. Compare these to simulations in Figures 3 and 4. The dotted lines show the apparent *Q*.

The *Q* of the inductor was measured and its series resistance was subtracted from the total series resistance to obtain the actual stub series resistance. This data was then used to correct the resistive part of the VNA measurements, by calculating the offset between the two measurements. This correction is only valid around the test frequency. Then the *Q* was computed using the corrected VNA measurements, with Equations 9 and 14.

Table 3 shows the calculated and measured *Q* factors at the (approximate) quarter wave frequency. The calculated values were obtained from the Equations 14 and 9, while the measured values were done by impedance measurements, using Equation 34. Keep in mind that the measured *Q* value for open line has probably more errors, since the resistive part of the impedance is around 1 Ω.

The conductor losses are dominant here and we can also use equation 16 to compute the resonator *Q* based on the measured stub attenuation in dB/100 ft. The *Q* value obtained is 32.00 at 19.375 MHz.

Using the Resonator as a Bandpass Filter

Knowing the unloaded *Q*, as calculated

Table 2

Shorted Line		Open Line	
Frequency (MHz)	*% Q error*	*Frequency (MHz)*	*% Q error*
9.33	2.7	4.92	2.9
28.93	2.1	28.93	1.5

Table 3

Shorted Line			Open Line		
Frequency (MHz)	*Calculated Q (Eq 14)*	*Measured Q*	*Frequency (MHz)*	*Calculated Q (Eq 9)*	*Measured Q*
19.375	35.3		19.375	36.36	
19.106		34.43	19.334		30.95

above, allows us to compute the insertion loss of a single resonator bandpass filter based on the percent bandwidth.

$$Loss = 20 \log \left[1 - \frac{100}{K \, Q_u} \right] \qquad \text{(Eq 35)}$$

Equation 35 shows the relation between the unloaded Q (Q_u), K the percentage bandwidth (ratio of bandwidth to center frequency in %) and the resonator losses in dB.[8] For instance, a 2% bandwidth with a Q_u of 200 will give an insertion loss of 2.5 dB.

Q Factor of Antennas

Equations 9 (for open circuit antennas) and 14 (for closed loop antennas) may also be used to calculate the Q factor of antennas, based on their impedance data. I have provided a *Mathcad* file that does impedance calculations on a monopole for various lengths.[7, 9] The monopole is treated as a transmission line whose average impedance Z_a is given by:

$$Z_a = 60 \ln (hd) \qquad \text{(Eq 36)}$$

where hd is the length-to-radius ratio. Here, the radiation resistance is proportional to the frequency to the power 1.7. It is added to the conductor losses.

Figure 14 gives the simulated monopole Q factor. Once the Q factor is known, the bandwidth, BW, may be easily calculated using Equation 37. The bandwidth obtained gives the frequencies where the impedance phase angle is ±45°.

$$BW = \frac{f_r}{Q} \qquad \text{(Eq 37)}$$

where f_r is the center frequency.

This bandwidth, as calculated from the Q factor, corresponds to the 7 dB return loss points or to an SWR of ~2.62. For a 2:1 SWR, the bandwidth is 70% of the above. This assumes that the SWR is 1:1 at resonance.

Conclusion

In this paper I have shown that a single general equation may be used to calculate the Q factor of RLC circuits and transmission lines as it relates to circuit selectivity. I have shown that series resonances are taken care by Equation 9 while Equation 14 takes care of parallel resonances. The difference between the two is the use of impedances for the first and admittances for the second equation. By using the provided *Mathcad* files and the associated public domain programs, it is easy to compute the Q factor of coaxial and microstrip or stripline resonators for any length and frequency. The calculated Q factor agrees very well with the measured data and with the Q values computed by the impedance measurement method. When simulating stub resonators, it is very important that the full R L G C transmission line models be used. The *Mathcad* files show how to optimize the Q factor of PCB microstrip and stripline resonators. The same files also show the calculations of R L C G functions and coefficients to be incorporated into any RF simulation software. It is interesting to note that these four frequency dependent parameters fully describe the transmission line. I found that the Q values predicted from Equations 9 and 14 fully agree with the values obtained in the simulation software.

The simulations presented here show that it is possible to get higher Q factors on low-loss lines by using higher-order modes, such as ¾ λ. For PCB resonator traces, the optimum Q is generally below a quarter wavelength.

Note that the models presented here do not take into account the radiation losses and surface roughness of microstrips. The line dc resistance has been included in the PCB line models, since its contribution is not negligible at narrow trace widths. For coaxial lines, the dc resistance has been omitted to keep the models simpler and relieve the user from searching for the resistance data. This makes the coaxial models less accurate below approximately 5 MHz. The dc resistance may be added to the ac conductor resistance by taking the square root of the sum of the squares of the dc and ac resistances, as done in the file: TRL_Q_Calc-PCB1.mcd.

Thanks to Chase Hearn and Yan Gunmar for triggering this study and providing various helpful references and comments.

Appendix

An intuitive view of the general function for calculating the Q factor is derived here.

The general equation for computing the Q factor of RLC circuits, transmission line resonators and antennas operating in the series resonant mode is given by the source at Note 1. This equation is valid from low frequencies, though series resonance and for frequencies much above series resonance.

$$Q = \frac{\omega}{2R} \left(\frac{dX}{d\omega} + \frac{|X|}{\omega} \right) \qquad \text{(Eq A1)}$$

Where $\omega = 2\pi f$, the radian frequency, R and X are respectively the real and imaginary parts of the impedance .

Equation A1 may be written in terms of the frequency:

$$Q = \frac{f}{2R} \left(\frac{dX}{df} + \frac{|X|}{f} \right) \qquad \text{(Eq A2)}$$
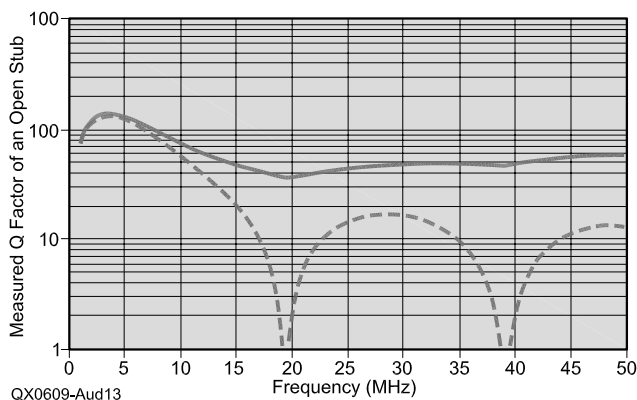
QX0609-Aud13

Figure 13 — Measured *Q* factor of an open line. Computed *Q* factor versus frequency (solid lines, from impedance measurements) for a 100 inch length of RG-58, giving first resonance at ~19.4 MHz. Compare these to simulations in Figures 3 and 4. The dotted lines show the apparent *Q*.
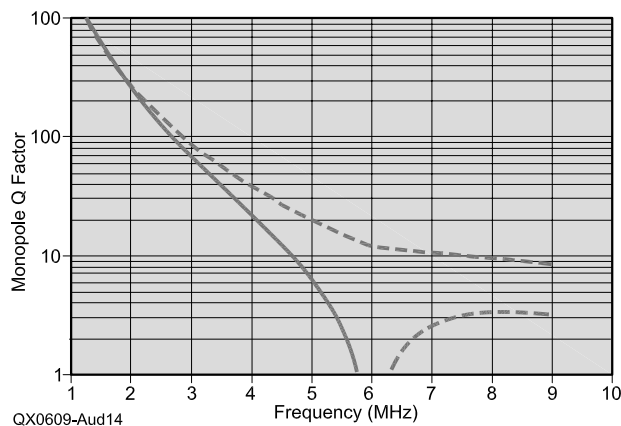
QX0609-Aud14

Figure 14 — Monopole *Q* factor (solid curve) as calculated from the *Mathcad* file: Monopole-Ralph.mcd. The quarter wave resonance is at 6 MHz, as shown by the dotted line curve (apparent *Q*) which was calculated from the reactance to resistance ratio. Note the steep increase in *Q* as the frequency is lowered. This comes from the fact that the radiation resistance decreases approximately as frequency to the power 1.7. At 1 MHz and below we have a low-loss air insulated capacitor!

Notice that the first term of Equation A2 represents the Q factor at resonance, when $X = 0$.

$$Q = \frac{f}{2R}\left(\frac{dX}{df}\right) \qquad \text{(Eq A3)}$$

This equation gives the Q Factor at resonance, as very often seen in textbooks.

Applying Equation A2 for a series resonant circuit, for frequencies well above the resonant frequency, we get:

$$\left(\frac{dX}{df}\right) = 2\pi L \qquad \text{(Eq A4)}$$

Recall that the reactive component comes from the inductance.

$$Q_{hf} = \frac{f}{2R}\left(2\pi L + \frac{|2\pi f L|}{f}\right) \qquad \text{(Eq A5)}$$

where Q_{hf} is the Q factor for frequencies well above the resonant frequency. This simplifies to:

$$Q_{hf} = \frac{2\pi f L}{R} = \frac{X_L}{R} \qquad \text{(Eq A6)}$$

where X_L is the inductive reactance. This is the same as Equation 1 in the main text — valid well above the resonant frequency.

For frequencies well below the resonant frequency, where the reactance component comes from the capacitance, we get:

$$\left(\frac{dX}{df}\right) = \frac{1}{2\pi f^2 C} \qquad \text{(Eq A7)}$$

$$Q_{lf} = \frac{f}{2R}\left(\frac{1}{2\pi f^2 C} + \frac{1}{2\pi f C f}\right) \qquad \text{(Eq A8)}$$

Q_{lf} is the Q factor for frequencies well below the resonant frequency.

After simplification:

$$Q_{lf} = \frac{1}{2\pi f R C} = \frac{X_C}{R} \qquad \text{(Eq A9)}$$

where X_C is the capacitive reactance — valid well below the resonant frequency. This is the same as Equation 1 in the main text.

Notes

[1]Peter Vizmuller, *RF Design Guide*, Artech House, Norwood, MA, 1995, p. 235.
[2]The Small Koch Fractal Monopole: Theory, design and applications, p 6. See **www-personal.engin. umich.edu/~lschulwi/koch.pdf**
[3]Professor Niknejad, Lectures on Lossy Transmission Lines and the Smith Chart, University of California, Berkeley. Course EECS 117, Lecture 6: **www-inst.eecs. berkeley.edu/~ee117/ sp04/lect/lecture6.pdf**
[4]The program TLDetails.exe may be downloaded from Dan Maguire, AC6LA. Download from **www.ac6la.com/tldetails.html** e-mail: **ac6la@arrl.net**
[5]The program TXLine.exe may be downloaded from **www.mwoffice.com/products/txline.html** (you will need to register.)
[6]Frank Witt, AI1H, "Transmission Line Properties from Manufacturer's Data," *ARRL Antenna Compendium Vol. 6*, ARRL, 1999, pp 179 - 183 and related *Mathcad* files on the accompanying CD-ROM.
[7]The *Mathcad* files TRL_Q_Calc1.mcd and TRL_Q_Calc-PCB1, Monopole-Ralph.mcd and the *Excel* file: Calc_SeriesRLC.xls may be downloaded from the ARRL Web at **www.arrl.org/qexfiles/**. Look for **9x06_ Audet.zip.**
[8]Randall W. Rhea, *HF Filter Design and Computer Simulation*, section 5.5, Noble Publishing.
[9]Ralph Holland, "Impedance of Wide Regular Structures," **antennex.com/archive3/ archive3.htm**. You must be an antenneX subscriber to view this article.

Jacques Audet, VE2AZX, became interested in radio at the age of 14, after playing with crystal radio sets and repairing old receivers. At 17, he obtained his first ham license. In 1967 he obtained his B Sc degree in electrical engineering from Laval University. He then worked in engineering functions at Nortel Networks, where he retired in 2000. He worked mostly in test engineering on a number of products and components operating from dc to light-wave frequencies. His areas of interest are in RF simulations, filters, duplexers, antennas and using computers to develop new test techniques in measurement and data processing. QEX‑